Lecture 26: Power Series

A **Power Series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

where x is a variable, the c_n's are constants called the coefficients of the series.

Example

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \dots$$

A power series may converge for some values of x and may diverge for others.

Example In the series above, if we replace x by 1, we get

$$\sum_{n=0}^{\infty} \frac{1^n}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

which converges and if we replace x by 2, we get

$$\sum_{n=0}^{\infty} \frac{2^n}{2^n} = 1 + \frac{2}{2} + \frac{2^2}{2^2} + \frac{2^3}{2^3} + \dots = 1 + 1 + 1 + 1 + 1 + \dots$$

which diverges.

A **power series defines a function**

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots$$

whose **domain** is the set of all values of x for which the series converges.

Example Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \dots$$

What is $f(0)$? What is the domain of f?

Definition A power series in $(x-a)$ or **a power series centered at** a is a power series of the form

$$\sum_{x=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

where c_n is a constant for all n.

Note that when $x = a$, we have

$$\sum_{x=0}^{\infty} c_n(x-a)^n = c_0 + c_1(a-a) + c_2(a-a) + c_3(a-a) + \cdots = c_0$$

and the series converges to c_0.
Note also that when $a = 0$, the power series about a above just becomes a power series about 0 similar to the power series in our original definition and the previous examples.

Example The power series below is centered at 1. Use the ratio test to determine the values of x for which the series converges

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(n+1)^3}$$

Theorem For any power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only 3 possibilities for the the values of x for which the series converges :

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Definition The **radius of convergence (R.O.C.)** of the power series
is the number R in case 3 above.
In case 1, the radius of convergence is 0 and
in case 2, the radius of convergence is ∞.

We see that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ always converges within some interval centered at a and diverges outside that interval. The **interval of convergence (I.O.C.)** of a power series is the interval that consists of all values of x for which the series converges.

- In case 1 above, the interval of convergence is a single point $\{a\}$.
- In case 2 above, the interval of convergence is $(-\infty, \infty)$.
- In case 3 above, the interval of convergence may be

$$(a - R, a + R), \quad [a - R, a + R), \quad (a - R, a + R], \quad [a - R, a + R].$$

Example Find the interval of convergence and radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$

Example Find the interval of convergence and radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1},$$

Example Find the interval of convergence and radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+1)4^n}$$

Extra Example Find the interval of convergence and radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} \frac{n(2x-1)^n}{5^n}$$

First we put this series in the correct form.

$$\sum_{n=0}^{\infty} \frac{n(2x-1)^n}{5^n} = \sum_{n=0}^{\infty} \frac{n(2[x-\frac{1}{2}])^n}{5^n} = \sum_{n=0}^{\infty} \frac{n2^n[x-\frac{1}{2}]^n}{5^n} =$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(n+1)2^{n+1}|x-\frac{1}{2}|^{n+1}}{5^{n+1}}\right|}{\left|\frac{(n)2^n|x-\frac{1}{2}|^n}{5^n}\right|} = \lim_{n\to\infty} \frac{2|x-\frac{1}{2}|}{5} \cdot \left(\frac{n+1}{n}\right) = \frac{2|x-\frac{1}{2}|}{5}.$$

The ratio test says that this power series converges if

$$\frac{2|x-\frac{1}{2}|}{5} < 1 \qquad \text{which is the same as} \qquad |x-\frac{1}{2}| < 5/2$$

Therefore the **radius of convergence** is $R = 5/2$. Since the power series diverges for values of x with $|x-\frac{1}{2}| > 5/2$, we determine the interval of convergence by checking if the series converges at the end points of the interval defined by this inequality.

$$|x-\frac{1}{2}| < 5/2 \quad \text{if} \quad -5/2 < x - 1/2 < 5/2 \quad \text{if} \quad -2 < x < 3.$$

$$\text{At} \quad x = -2, \quad \sum_{n=0}^{\infty} \frac{n(2x-1)^n}{5^n} = \sum_{n=0}^{\infty} \frac{n(-5)^n}{5^n} = \sum_{n=0}^{\infty} n(-1)^n$$

which <u>diverges</u> since $\lim_{n\to\infty} n \neq 0$.

$$\text{At} \quad x = 3, \quad \sum_{n=0}^{\infty} \frac{n(2x-1)^n}{5^n} = \sum_{n=0}^{\infty} \frac{n(5)^n}{5^n} = \sum_{n=0}^{\infty} n$$

which <u>diverges</u> since $\lim_{n\to\infty} n \neq 0$.
Therefore the **interval of convergence** is $(-2, 3)$.