Power Series

Definition A **Power Series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

where x is a variable, the c_n's are constants called the coefficients of the series.

Example We already know a lot about the power series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$.

- When $x = 0$, this power series is the series $\sum_{n=0}^{\infty} 0^n = 1 + 0 + 0 + \dots$ which converges to 1.
- When $x = \frac{1}{4}$, this power series becomes $\sum_{n=0}^{\infty} \frac{1}{4^n} = 1 + \frac{1}{4} + \frac{1}{4^2} + \dots$ which converges to $\frac{1}{1-1/4} = 4/3$.
- When $x = 2$, this power series becomes $\sum_{n=0}^{\infty} 2^n = 1 + 2 + 2^2 + \dots$ which diverges since it is a geometric series with $r = 2 > 1$.
- We see that a power series can converge for some values of x and diverge for others.
- In this case the series $\sum_{n=0}^{\infty} x^n$ converges if $|x| < 1$ and diverges if $|x| \geq 1$, since it is a geometric series with $r = x$.

Example: $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

Example Lets see what we can say about the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \dots$$

- When $x = 0$, this power series is the series $\sum_{n=0}^{\infty} 0^n = 1 + 0 + 0 + \dots$ which converges to 1.
- When $x = 1$, this power series becomes $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$ which converges to $\frac{1}{1-1/2} = 2$.
- When $x = 2$, this power series becomes $\sum_{n=0}^{\infty} \frac{2^n}{2^n} = 1 + 1 + 1 + \dots$ which diverges since it is a geometric series with $r = 1 > 1$.
- Note $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$. This series is always a geometric series with $r = \frac{x}{2}$.
- Therefore this series converges when $\left|\frac{x}{2}\right| < 1$ and diverges if $\left|\frac{x}{2}\right| \geq 1$.
- Therefore this series converges when $|x| < 2$ and diverges if $|x| \geq 2$.
- Since this series converges only on the interval $-2 < x < 2$, we say that the **Interval of Convergence** of this series is the interval $(-2, 2)$.

Power series as functions

A **power series defines a function**

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots$$

whose **domain** is the set of all values of x for which the series converges.

Example Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \dots$
What is $f(0)$? What is the domain of f?

- $f(0) = 1 + 0 + 0 + 0 + \cdots = 1.$
- $f(1) = \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{1}{1-(1/2)} = 2.$
- The domain of f is all x for which $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converges.
- Therefore the domain of f is all x in the interval $-2 < x < 2$.
- In fact for every value of x in this interval, we can find a formula for $f(x)$ using our knowledge of geometric series.
- For $-2 < x < 2$, $f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \frac{1}{1-(x/2)} = \frac{2}{2-x}$.
- We cannot always get a formula for a function defined by a power series. We will focus on that problem in subsequent lectures. Today we focus on finding the domain (Interval of convergence) of a power series.

Power Series Centered at a.

Definition A power series in $(x-a)$ or **a power series centered at** a is a power series of the form

$$\sum_{x=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

where c_n is a constant for all n.

- **Note** that when $x = a$, we have

$$\sum_{x=0}^{\infty} c_n(x-a)^n = c_0 + c_1(a-a) + c_2(a-a) + c_3(a-a) + \dots = c_0$$

  and the series converges to c_0.

- **Note** also that when $a = 0$, the power series about a above just becomes a power series about 0 similar to the power series in our original definition and the previous examples.
- That is the power series $\sum_{x=0}^{\infty} \frac{x^n}{2^n}$ is a power series centered around $a = 0$.

Example

Example The power series below is centered at 1. Use the ratio test to determine the values of x for which the series converges $\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(n+1)^3}$

- For any given value of x, we apply the ratio test to the series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(n+1)^3}$.
- We have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-1|^{n+1} \Big/ (3^{n+1}(n+2)^3)}{|x-1|^n \Big/ (3^n(n+1)^3)}$
$= \lim_{n\to\infty} \frac{|x-1|^{n+1}}{|x-1|^n} \frac{3^n(n+1)^3}{3^{n+1}(n+2)^3} = \lim_{n\to\infty} \frac{|x-1|^{n+1}}{|x-1|^n} \frac{3^n}{3^{n+1}} \frac{(n+1)^3}{(n+2)^3}$
$= \lim_{n\to\infty} \frac{|x-1|}{3} \left(\frac{n+1}{(n+2)}\right)^3 = \frac{|x-1|}{3} \lim_{n\to\infty} \left(\frac{n+1}{(n+2)}\right)^3 = \frac{|x-1|}{3}$.
- Since we are using the ratio test, we conclude that the series converges if $\frac{|x-1|}{3} < 1$, it diverges if $\frac{|x-1|}{3} > 1$ and the test is inconclusive when $\frac{|x-1|}{3} = 1$.
- That is, the series converges if $|x-1| < 3$, that is $-3 < x-1 < 3$. Adding 1 to both sides, we see this is equivalent to $-2 < x < 4$.
- The series diverges if $\frac{|x-1|}{3} > 1$ or $|x-1| > 3$, that is $x-1 > 3$ or $x-1 < -3$. Therefore the series diverges when $x < -2$ or $x > 4$.
- The test is inconclusive when $\frac{|x-1|}{3} = 1$, that is when $x = -2$ or $x = 4$.

Example continued

Example The power series below is centered at 1. Use the ratio test to determine the values of x for which the series converges $\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(n+1)^3}$

- We concluded the series converges if $-2 < x < 4$.
- The series diverges if $x < -2$ or $x > 4$.
- The test is inconclusive when $x = -2$ or $x = 4$.
- We treat these two cases separately.
- When $x = -2$, the series becomes $\sum_{n=0}^{\infty} \frac{(-2-1)^n}{3^n(n+1)^3} = \sum_{n=0}^{\infty} \frac{(-3)^n}{3^n(n+1)^3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^3}$. This series converges absolutely, since $\sum_{n=0}^{\infty} \frac{1}{(n+1)^3}$ converges by the limit comparison test, comparing with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.
- When $x = 4$, the series becomes $\sum_{n=0}^{\infty} \frac{(4-1)^n}{3^n(n+1)^3} = \sum_{n=0}^{\infty} \frac{(3)^n}{3^n(n+1)^3} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^3}$ which converges by the limit comparison test, comparing with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.
- Therefore, this series converges for x in the closed interval $[-2, 4]$ and diverges otherwise.

Radius of Convergence, Interval of Convergence

Theorem For any power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only 3 possibilities for the the values of x for which the series converges :

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. (In previous example $R = 3$, series converged when $|x-1| < 3$ and diverged when $|x-1| > 3$.)

- **Definition** The **Radius of convergence (R.O.C.)** of the power series is the number R in case 3 above.
  In case 1, the R.O.C. is 0 and in case 2, the R.O.C. is ∞.
- We see that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ always converges within some interval centered at a and diverges outside that interval. The **Interval of Convergence (I.O.C.)** of a power series is the interval that consists of all values of x for which the series converges.
  - In case 1 above, the interval of convergence is a single point $\{a\}$,
  - In case 2 above the interval of convergence is $(-\infty, \infty)$.
  - In case 3 above the interval of convergence may be

$$(a-R, a+R), \quad [a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R].$$

Previous Example

Example The power series below is centered at 1.

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(n+1)^3}.$$

We used the ratio test to determine that the series converges when $|x-1| < 3$ and diverges when $|x-1| > 3$. Therefore the radius of convergence of this series is 3.
We checked the endpoints of the interval $(-2, 4)$ using the limit comparison test to find that the series converges on the interval $[-2, 4]$ and diverges otherwise.
Therefore the interval of convergence for this series is $[-2, 4]$.

Example

Example Find the interval of convergence and radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$

- We use the ratio test to determine where the series converges.
- $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \frac{n!}{(n+1)!}$
$= \lim_{n\to\infty} \frac{|x|}{(n+1)} = |x| \lim_{n\to\infty} \frac{1}{(n+1)} = 0$
- This limit is zero for every value of x. This means that the power series converges for every value of x. Here the **radius of convergence** is $R = \infty$ and the **Interval of Convergence** is $(-\infty, \infty)$.

Example

Example Find the interval of convergence and radius of convergence of the following power series: $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1}$.

- $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}x^{n+1}}{2(n+1)+1}\right|}{\left|\frac{(-1)^n x^n}{2n+1}\right|} = \lim_{n\to\infty} \frac{|x|(2n+1)}{(2n+3)} =$ $|x| \lim_{n\to\infty} \frac{(2n+1)}{(2n+3)} = |x|$.
- By the ratio test,this series converges if $|x| < 1$ and diverges if $|x| > 1$. The **Radius of Convergence** is $R = 1$.
- To determine the I.O.C., we check the endpoints of the interval $|x| < 1$ or $-1 < x < 1$ giving x in $(-1, 1)$.
- At $x = 1$: $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 1^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. This converges by the alternating series test, since $\lim_{n\to\infty} \frac{1}{2n+1} = 0$ and $\frac{1}{2n+1} > \frac{1}{2(n+1)+1}$ for all $n > 1$.
- At $x = -1$: $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{2n+1}$. This series diverges by comparison to the harmonic series $\sum \frac{1}{n}$ which is known to diverge. We have $\lim_{n\to\infty} \frac{1/(2n+1)}{1/n} = 1/2$; Both diverge.
- Therefore the **Interval of Convergence** for this power series is $(-1, 1]$.

Example

Example Find the interval of convergence and radius of convergence of the following power series: $\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+1)4^n}$.

- $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(x+2)^{n+1}}{(n+2)4^{n+1}}\right|}{\left|\frac{(x+2)^n}{(n+1)4^n}\right|} = \lim_{n\to\infty} \frac{|x+2|\cdot(n+1)}{(n+2)\cdot 4} = \frac{|x+2|}{4} \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right) = \frac{|x+2|}{4}$.
- The ratio test says that this power series converges if $\frac{|x+2|}{4} < 1$ or $|x+2| < 4$ and the series diverges if $|x+2| > 4$. The **Radius of Convergence** of this power series is $R = 4$.
- To determine the I.O.C., we check the endpoints of the interval $|x+2| < 4$ or $-4 < x+2 < 4$ giving x in $(-6, 2)$.
- At $x = 2$: $\sum_{n=0}^{\infty} \frac{(2+2)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(4)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ which diverges by (limit) comparison with the harmonic series.
- At $x = -6$: $\sum_{n=0}^{\infty} \frac{(-6+2)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(-4)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ which converges by the alternating series test.
- Therefore the **Interval of Convergence** of this series is $[-6, 2)$.