Power Series representations of Functions

From our knowledge of Geometric Series, we know that

$$g(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad |x| < 1.$$

- Recall that we had

  $$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if} \quad -1 < r < 1$$

  and this series diverges for $|r| \geq 1$.
  Above we have $a = 1$ and $x = r$.

This gives us **a power series representation for the function $g(x)$ on the interval $(-1, 1)$.**
Note that the function $g(x)$ here has a larger domain than the power series.

- The n th partial sum of the above power series is given by $P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$.
- Hence, as $n \to \infty$, the graphs of the polynomials, $P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$ get closer to the graph of $f(x)$ on the interval $(-1, 1)$.

Questions of the day

Q1. Given a function, can I find a power series representation of that function.

- Example: Can I find a power series representation for the function $f(x) = \frac{1}{1+x^7}$ or $h(x) = \ln(1+x)$.

Part of the same question: If so for which values of x is the power series representation valid?

- Suppose I know that $\frac{1}{1+x^7} = 1 - x^7 + x^{14} - x^{21} + x^{28} - \cdots + (-1)^n x^{7n} \cdots = \sum_{n=0}^{\infty} (-1)^n x^{7n}$, for some values of x, for which values of x is this power series representation valid?

Q2 Why on earth would I want to write a nice function like that as a P.S.?

- The formula $\frac{1}{1+x^7} = 1 - x^7 + x^{14} - x^{21} + x^{28} - \cdots + (-1)^n x^{7n} \cdots = \sum_{n=0}^{\infty} (-1)^n x^{7n}$ allows me to actually calculate the sums of a new set of series e.g. I know that $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{7n}} = \frac{1}{1+\frac{1}{2^7}}$.
- Also I can approximate the function $\frac{1}{1+x^7}$ when $|x| < 1$ by the polynomial $1 - x^7 + x^{14}$.
- More importantly I can approximate an antiderivative of this function $\int \frac{1}{1+x^7} dx$ by a polynomial $x - \frac{x^8}{8} + \frac{x^{15}}{15}$.

Deriving new representations from old ones

Substitution First, we examine how to use the power series representation of the function $g(x) = 1/(1-x)$ on the interval $(-1, 1)$ to derive a power series representation of other functions on an interval.

Example Find a power series representation of the function given below and find the interval of convergence of the series. $f(x) = \frac{1}{1+x^7}$.

- We have $\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots = \sum_{n=0}^{\infty} y^n \quad \text{for} \quad -1 < y < 1$
- Now $\frac{1}{1+x^7} = \frac{1}{1-(-x^7)}$.
- Substituting $-x^7$ for y in the above equation, we get $\frac{1}{1-(-x^7)} = 1 + (-x^7) + (-x^7)^2 + (-x^7)^3 + \cdots = \sum_{n=0}^{\infty} (-x^7)^n \quad \text{for} \quad -1 < (-x^7) < 1$
- or

$$\frac{1}{1+x^7} = 1 - x^7 + x^{2(7)} - x^{3(7)} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{7n} \quad \text{for} \quad -1 < x < 1$$

  since we have $-1 < -x^7 < 1$ if $1 > x^7 > -1$ or $-1 < x^7 < 1$ or $-1 < x < 1$.

- This is the interval of convergence of the power series $\sum_{n=0}^{\infty} (-1)^n x^{7n}$, since this series is easily seen to diverge at $x = 1$ and $x = -1$..

Example (Substitution)

Example Find a power series representation of the function given below and find the interval of convergence of the series.

$$f(x) = \frac{2x}{1+x}.$$

- As in the previous example, we have that $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots = \sum_{n=0}^{\infty} (-x)^n$ for $-1 < (-x) < 1$
- So $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$ for $-1 < x < 1$.
- Since $f(x) = \frac{2x}{1+x} = 2x \frac{1}{1+x}$, we have
- $f(x) = 2x\left[1 - x + x^2 - x^3 + \cdots\right] = \left[2x1 - 2xx + 2xx^2 - 2xx^3 + \cdots\right]$.
- $= 2x - 2x^2 + 2x^3 - 2x^4 + \cdots$.
- Alternatively in shorthand:

$$f(x) = \frac{2x}{1+x} = 2x \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} 2x(-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n 2x^{n+1}.$$

Example (Substitution)

Example Find a power series representation of the function given below and find the interval of convergence of the series $h(x) = \frac{1}{4-x}$

- We would like to use our knowledge of the well known geometric series $\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots + y^n + \cdots = \sum_{n=0}^{\infty} y^n \quad \text{for} \quad -1 < y < 1$
- $h(x) = \frac{1}{4-x} = \frac{1}{4}\left[\frac{1}{1-\frac{x}{4}}\right]$.
- With $y = \frac{x}{4}$, we get $\frac{1}{1-\left(\frac{x}{4}\right)} = 1 + \left(\frac{x}{4}\right) + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{4}\right)^3 + \cdots + \left(\frac{x}{4}\right)^n + \cdots = \sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n \quad \text{for} \quad -1 < \left(\frac{x}{4}\right) < 1$
- Therefore $\frac{1}{1-\left(\frac{x}{4}\right)} = 1 + \frac{x}{4} + \frac{x^2}{4^2} + \frac{x^3}{4^3} + \cdots + \frac{x^4}{4^n} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{4^n} \quad \text{for} \quad -4 < x < 4$
- It is clear that this series diverges at the endpoints, so the interval of convergence of the series is $(-4, 4)$.
- We have $h(x) = \frac{1}{4}\left[\frac{1}{1-\frac{x}{4}}\right] =$
  $\frac{1}{4}[1 + \frac{x}{4} + \frac{x^2}{4^2} + \frac{x^3}{4^3} + \cdots + \frac{x^4}{4^n} + \cdots] = \frac{1}{4}\sum_{n=0}^{\infty} \frac{x^n}{4^n}$
  $= \frac{1}{4} + \frac{x}{4^2} + \frac{x^2}{4^3} + \frac{x^3}{4^4} + \cdots + \frac{x^4}{4^{n+1}} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{4^{n+1}}$

Differentiation and Integration of Power Series

We can differentiate and integrate power series term by term, just as we do with polynomials.

Theorem If the series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$

Also

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}.$$

The radii of convergence of both of these power series is also R. (The interval of convergence may not remain the same when a series is differentiated or integrated; in particular convergence or divergence may change at the end points).

Example; Differentiation

Example Find a power series representation of the function $\frac{1}{(x+1)^2}$.

- Above we found that
  $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^n \quad \text{for} \quad -1 < x < 1$
- Therefore we have
  $\frac{d}{dx}\left[\frac{1}{1+x}\right] = \frac{d}{dx}\left[1-x+x^2-x^3+\cdots\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$ for $-1 < x < 1$
- Differentiating we get
  $\frac{-1}{(1+x)^2} = 0 - 1 + 2x - 3x^2 + \cdots + (-1)^n n x^{n-1} + \cdots = \sum_{n=0}^{\infty} \frac{d}{dx}(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^n n x^{n-1} = \sum_{n=1}^{\infty}(-1)^n n x^{n-1}$ when $-1 < x < 1$.
- Since the series has the same radius of convergence when differentiated, we know that this new series converges on the interval $-1 < x < 1$.
- (Note we can set the limits of the new sum from n $=$ 0 to infinity if we like, since that just gives an extra 0 at the beginning or we can drop the n=0 term; this is merely a cosmetic change.)
- Now we multiply both sides by -1 to get
  $\frac{1}{(1+x)^2} = 0 + 1 - 2x + 3x^2 + \cdots + (-1)^{n+1} n x^{n-1} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} \quad \text{for} \quad -1 < x < 1$

Example; Integration

Example Find a power series representation of the function $\ln(1+x)$.

- Above, we showed that
  $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^n \quad \text{for} \quad -1 < x < 1$.
- We have $\int \frac{1}{1+x} dx = \int [1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots] dx =$
  $\int \sum_{n=0}^{\infty}(-1)^n x^n dx \quad \text{for} \quad -1 < x < 1$
- integrating the left hand side and integrating the right hand side term by term, we get
  $\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots =$
  $\sum_{n=0}^{\infty} \int (-1)^n x^n dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + C$ for $-1 < x < 1$.
- To find the appropriate constant term, we let $x = 0$ in this equation. We get
  $$\ln(1+0) = C + 0 - 0 + 0 - 0 + \cdots = C$$
- Therefore $C = 0$ and
  $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots =$
  $\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} \quad \text{for} \quad -1 < x < 1$

Example; Approximation

Example Find an approximation of $\ln(1.1)$ with error less than 10^{-5}.

- In the previous example, we saw that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \text{for} \quad -1 < x < 1$
- Using $x = 0.1$, we get $\ln(1.1) = 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \cdots + (-1)^n \frac{(0.1)^{n+1}}{n+1} + \cdots$.
- Recall that if we have an alternating series which converges by the Alternating series test (the above series does), then if we estimate the sum of the series by adding the first M terms, the error $\left| \ln(1.1) - \sum_{n=0}^{M} (-1)^n \frac{(0.1)^{n+1}}{n+1} \right| =$ $\left| \sum_{n=0}^{\infty} (-1)^n \frac{(0.1)^{n+1}}{n+1} - \sum_{n=0}^{M} (-1)^n \frac{(0.1)^{n+1}}{n+1} \right| \le |a_{M+1}|$
- Therefore, if we use the estimate $\ln(1.1) \approx \sum_{n=0}^{3} (-1)^n \frac{(0.1)^{n+1}}{n+1} = 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} =$ $0.1 - 0.005 + 0.000333 - 0.000025 = 0.0953083$,
  we must have an error less than or equal to $|a_{M+1}| = \left| \frac{(0.1)^5}{5} \right| < 10^{-5}$.
- If we use a computer to check, we see that $|\ln(1.1) - 0.0953083| = 1.8798 \times 10^{-}6$.

Approximation of definite Integrals

Example Use power series to approximate the following integral with an error less than 10^{-10}: $\int_0^{0.1} \frac{1}{1+x^7}\,dx$.

- By substituting $-x^7$ for x in the power series representation of $1/(1-x)$, we got for $-1 < x < 1$
  $\frac{1}{1+x^7} = 1 - x^7 + x^{14} - x^{21} + x^{28} - \cdots + (-1)^n x^{7n} \cdots = \sum_{n=0}^{\infty}(-1)^n x^{7n}$.
- Now taking the integral of both sides, we get
  $\int_0^{0.1} \frac{1}{1+x^7}\,dx = \int_0^{0.1}[1 - x^7 + x^{14} - x^{21} + x^{28} - \cdots + (-1)^n x^{7n} \cdots]dx =$
  $\left[x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots + (-1)^n \frac{x^{7n+1}}{7n+1} \cdots\right]_0^{0.1}$
  $$= (.1) - \frac{(.1)^8}{8} + \frac{(.1)^{15}}{15} - \frac{(.1)^{22}}{22} + \cdots + (-1)^n \frac{(.1)^{7n+1}}{7n+1} \cdots$$
- This is an alternating series which sums to the definite integral $\int_0^{0.1} \frac{1}{1+x^7}\,dx$. I can estimate the sum of the series by taking a partial sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n$ and the error of approximation is less than or equal to $|a_{n+1}|$.
- Since $\left|\frac{(.1)^{15}}{15}\right| < (.1)^{15} < (.1)^{10}$, we must have that $\left|\int_0^{0.1} \frac{1}{1+x^7}\,dx\right| \approx (.1) - \frac{(.1)^8}{8} = .1000000125$ and the error of approximation is less than 10^{-10}.

Example; Integration

Example Find a power series representation of the function $\tan^{-1}(x) = \int \frac{1}{1+x^2}dx$ and use your answer calculate $\sum_{n=0}^{\infty}(-1)^n \frac{1}{\sqrt{3}^{(2n+1)}(2n+1)}$.

- Using the power series representation of $\frac{1}{1-x}$ on the interval $(-1, 1)$, we get a power series representation of $\frac{1}{1+x^2}$:
  $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots$ on the interval $(-1, 1)$.
- Now we can integrate term by term to get a power series representation of $\tan^{-1}(x)$ on the interval $(-1, 1)$,
  $$\tan^{-1}(x) = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots \quad \text{for} \quad x \in (-1, 1).$$
- Since $\tan^{-1}(0) = 0$, we have $C = 0$ and
  $$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots \quad \text{on the interval} \quad (-1, 1).$$
- $\frac{1}{\sqrt{3}} < 1$ and with $x = \frac{1}{\sqrt{3}}$, we get
  $$\frac{\pi}{6} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{\sqrt{3}^{(2n+1)}(2n+1)}.$$