Lecture 28/29: Taylor Series and MacLaurin series

We saw last time that some functions are equal to a power series on part of their domain. For example

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad = \sum_{n=0}^{\infty} x^n, \quad \text{for} \quad -1 < x < 1,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \text{for} \quad -1 < x < 1,$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots \quad \text{on the interval} \quad (-1, 1).$$

In this section, we will develop a method to find power series expansions/representations for a wider range of functions and devise a method to identify the values of x for which the function equals the power series expansion. (This is not always the entire interval of convergence of the power series.)

Definition We say that $f(x)$ **has a power series expansion** at a if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for all} \quad x \ \text{ such that } \ |x-a| < R$$

for some $R > 0$

Example We see that $f(x) = \frac{1}{1-x}$, $g(x) = \ln(1+x)$ and $h(x) = \tan^{-1} x$ all have powers series expansions at 0.

Sometimes a function has a power series expansion at a point a and sometimes it does not. We saw some of the benefits of the existence of such an expansion in the last lecture. Before finding power series expansions of some well known functions we will examine the questions

- **Q1.** If a function $f(x)$ has a power series expansion at a, can we tell what that power series expansion is?
- **Q2.** For which values of x do the values of $f(x)$ and the sum of the power series expansion coincide? Note that this is a *different* question from asking what the IOC of the power series is.

Taylor Series

Definition If $f(x)$ is a function with infinitely many derivatives at a, the **Taylor Series** of the function $f(x)$ at/about a is the power series

$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots}$$

If $a = 0$ this series is called the **MacLaurin Series** of the function f:

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \cdots}$$

Derivatives of Taylor series of f match the derivatives of f at a

The Taylor series of f at a is given by

$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots},$$

If $T(x)$ is defined in an open interval around a, then it is differentiable at a, since it is a power series. Furthermore, every derivative of $T(x)$ at a equals the corresponding derivative of $f(x)$ at a.

$$T'(x) = 0 + f'(a) + \frac{2f^{(2)}(a)}{2!}(x-a) + \frac{3f^{(3)}(a)}{3!}(x-a)^2 + \ldots$$

$$T''(x) = 0 + 0 + \frac{2!f^{(2)}(a)}{2!} + \frac{3 \cdot 2 \cdot f^{(3)}(a)}{3!}(x-a) + \ldots$$

$$T^{(3)}(x) = 0 + 0 + 0 + \frac{3!f^{(3)}(a)}{3!} + \ldots etc....$$

So

$$T(a) = f(a) + 0 + 0 + \cdots = f(a)$$

$$T'(a) = f'(a) + 0 + 0 + \cdots = f'(a)$$

$$T''(a) = \frac{2!f^{(2)}(a)}{2!} + 0 + 0 + \cdots = f^{(2)}(a)$$

$$T^{(3)}(a) = \frac{3!f^{(3)}(a)}{3!} + 0 + \cdots = f^{(3)}(a)$$

Example Find the MacLaurin Series of the function $f(x) = e^x$. Find the radius of convergence of this series.

Important Limit Last class, we showed that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for any value of x. Therefore, we can conclude that

$$\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \text{for all values of} \quad x.}$$

Example Find the MacLaurin Series of the function $f(x) = \sin x$. Find the radius of convergence of this series.

Example Find the Taylor series expansion of the function $f(x) = e^x$ at $a = 1$. Find the radius of convergence of this series.

Answer to Q1

Theorem If f has a power series expansion at a, that is if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for all} \quad x \text{ such that } |x-a| < R$$

for some $R > 0$, then that power series is the Taylor series of f at a. We must have

$$c_n = \frac{f^{(n)}(a)}{n!} \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all $\quad x$ such that $|x-a| < R$.

If $a = 0$ the series in question is the MacLaurin series of f.

Example This result is saying that **if** $f(x) = e^x$ has a power series expansion at 0, then that power series expansion must be the MacLaurin series of e^x which is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

However the result is **not saying** that e^x sums to this series for all x. For that we need Taylor's theorem below.

Example The result also says that if $f(x) = e^x$ has a power series expansion at 1, then that power series expansion must be

$$e + e(x-1) + \frac{e(x-1)^2}{2!} + \frac{e(x-1)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!}$$

Q2: When does $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$?

Finding the values of x for which the Taylor series of a function $f(x)$ converges to $f(x)$.

For any value of x, the Taylor series of the function $f(x)$ about $x = a$ converges to $f(x)$ when the partial sums of the series ($T_n(x)$ below) converge to $f(x)$. We let

$$\boxed{R_n(x) = f(x) - T_n(x)},$$

where

$$\boxed{T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.}$$

$T_n(x)$ given above is called **the nth Taylor polynomial of f at a** and $R_n(x)$ is called the **remainder** of the Taylor series.

Theorem Let $f(x)$, $T_n(x)$ and $R_n(x)$ be as above. If

$$\lim_{n\to\infty} R_n(x) = 0 \quad \text{for} \quad |x-a| < R,$$

then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

To help us determine $\lim_{n\to\infty} R_n(x)$, we have the following inequality:

Taylor's Inequality If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$ then the remainder $R_n(x)$ of the Taylor Series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for} \quad |x-a| \le d.$$

Example: Taylor's Inequality applied to $\sin x$. If $f(x) = \sin x$, then for any n, $f^{(n+1)}(x)$ is either $\pm\sin x$ or $\pm\cos x$. In either case $|f^{(n+1)}(x)| \le 1$ for all values of x. Therefore, with $M = 1$ and $a = 0$ and d any number, Taylor's inequality tells us that $|R_n(x)| \le \frac{1}{(n+1)!}|x|^{n+1} \quad$ for $\;|x| \le d$.

Example: Taylor's Inequality applied to e^x. If $h(x) = e^x$, then for any value of n, $h^{(n+1)}(x) = e^x$. Now if d is any number, I know that $|h^{(n+1)}(x)| = |e^x| < e^d$ for all x with $|x| < d$. Hence applying Taylor's inequality to the MacLaurin series for e^x (with $a = 0$) we get that $|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1} \quad$ for $\;|x| \le d$.

Example Prove that $\sin x$ is equal to the sum of its MacLaurin series for all x, that is, show that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

for all x.

Example Prove that e^x is equal to the sum of its MacLaurin series for all x, that is, show that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

for all x.

Example Find the sum of the series $\sum_{n=0}^{\infty} \frac{2^n}{n!}$.

Example Find a power series representation for $\cos x$

Extra Applications

Using Taylor series to evaluate limits By expressing a function in terms of its Taylor series, you can sometimes evaluate limits that would be hard/tedious to do using L'Hopital's rule.
Example Use power series to find the limit

$$\lim_{x\to 0} \frac{\cos(x^5) - 1}{x^{10}}$$

As in the previous section, we can use known power series representations of functions to derive power series representations of related functions by substitution, differentiation or integration. Below, we show a table of the most commonly used series.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Example Find a power series representation for e^{-x^2}.

We know that

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots + \frac{y^n}{n!} + \ldots$$

Therefore

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \cdots + \frac{(-x^2)^n}{n!} + \ldots$$

$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \ldots$$

Example Use our results about alternating series to estimate $\int_0^1 e^{-x^2}dx$ with an error less than .001.

From above, we have that

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \ldots$$

Integrating term by term, we get

$$\int e^{-x^2}dx = C + x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot n!}.$$

Therefore

$$\int_0^1 e^{-x^2}dx = 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \cdots + \frac{(-1)^n}{(2n+1)\cdot n!}.$$

If we use the n th partial sum to approximate the integral, then since the series is alternating and converges by the A.S.T., we have an error $\leq |a_{n+1}| = \frac{1}{(2n+3)\cdot(n+1)!}$.
Therefore, if we choose n so that $\frac{1}{(2n+3)\cdot(n+1)!} < .001 = \frac{1}{1000}$, we know that the error of our approximation is $< .001$.
We have $\frac{1}{(2n+3)\cdot(n+1)!} < \frac{1}{1000}$ if $1000 < (2n+3)\cdot(n+1)!$. By trial and error, we get that this is true if $n = 4$. Therefore we have

$$\left|\int_0^1 e^{-x^2}dx - [1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!}]\right| < .001$$

that is

$$\left|\int_0^1 e^{-x^2}dx - [1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!}]\right| < .001$$

Therefore

$$\left|\int_0^1 e^{-x^2}dx - 0.747487\right| < .0001.$$

Example What is the MacLaurin series for the function $f(x) = \sqrt{x+1}$?

$$f(x) = (x+1)^{1/2},\quad f'(x) = \frac{1}{2}(x+1)^{-1/2},\quad f''(x) = \frac{1}{2}(\frac{-1}{2})(1+x)^{-3/2},\quad f^{(3)}(x) = \frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})(1+x)^{-5/2}$$

$$f(0) = 1,\quad f'(0) = \frac{1}{2},\quad f''(0) = \frac{1}{2}(\frac{-1}{2}),\quad f^{(3)}(1) = \frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})$$

$$f^{(n)}(0) = \frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})\ldots(\frac{1}{2} - (n+1)).$$

$$\frac{f^{(n)}(0)}{n!} = \frac{\frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})\ldots(\frac{1}{2} - (n+1))}{n!} = \binom{\frac{1}{2}}{n}.$$

We get

$$(1+x)^{1/2} = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}x^n.$$

For any real number k, let

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}.$$

Theorem : Binomial series If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Note This is just the binomial theorem if k is a positive integer.

An Example where $f(x)$ = McL series only at $x = 0$, but the McL series converges for all x
Example The function

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

turns out to have infinitely many derivatives at $a = 0$ and hence has a MacLaurin series

$$0 + 0x + 0x^2 + \cdots = 0 \quad \text{for all values of } x.$$

So we see that the MacLaurin series converges here for all values of x, but its sum does not equal the value of $f(x)$ for any x other than 0, because $e^{-1/x^2} > 0$ for all $x \neq 0$. In the graph below, the series is shown in red and $f(x)$ in blue.