So Far

We saw last day that some functions are equal to a power series on part of their domain. For example

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad \text{for} \quad -1 < x < 1,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \text{for } -1 < x < 1,$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots \quad \text{on the interval} \quad (-1, 1).$$

In this section, we will develop a method to find power series expansions/representations for a wider range of functions and devise a method to identify the values of x for which the function equals the power series expansion. (This is not always the entire interval of convergence of the power series.)

Definition

Definition We say that $f(x)$ **has a power series expansion** at a if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for all} \quad x \text{ such that } |x-a| < R$$

for some $R > 0$

Note $f(x)$ has a power series expansion at 0 if

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \text{for all} \quad x \text{ such that } |x| < R$$

for some $R > 0$.

Example We see that $f(x) = \frac{1}{1-x}$, $g(x) = \ln(1+x)$ and $h(x) = \tan^{-1} x$ all have powers series expansions at 0.

Questions

Sometimes a function has a power series expansion at a point a and sometimes it does not. One of the benefits of the existence of such an expansion is that we can approximate values of the function with a polynomial. Another is that we can actually find the sum of some series.
Our main questions are

- **Q1.** If a function $f(x)$ has a power series expansion at a, can we tell what that power series expansion is?
- **Q2.** For which values of x do the values of $f(x)$ and the sum of the power series expansion coincide?
- We will see that in answer to question 1, we can give a precise formula for the power series.
- We will examine the error in estimation by partial sums to answer question 2.

Taylor and McLaurin Series

Definition If $f(x)$ is a function with infinitely many derivatives at a, the **Taylor Series** of the function $f(x)$ at/about a is the power series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

If $\boxed{a = 0}$ this series is called the **McLaurin Series** of the function f:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

Matching derivatives

The Taylor series of f at a is given by $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$.

- If $T(x)$ is defined in an open interval around a, then it is differentiable at a, since it is a power series.
- Furthermore, every derivative of $T(x)$ at a equals the corresponding derivative of $f(x)$ at a.
- by changing x to a in the formula above, we see that $T(a) = f(a) + 0 + 0 + \cdots = f(a)$.
- $T'(x) = 0 + f'(a) + \frac{2f^{(2)}(a)}{2!}(x-a) + \frac{3f^{(3)}(a)}{3!}(x-a)^2 + \ldots$, So $T'(a) = f'(a) + 0 + 0 + \cdots = f'(a)$.
- $T''(x) = 0 + 0 + \frac{2!f^{(2)}(a)}{2!} + \frac{3 \cdot 2 \cdot f^{(3)}(a)}{3!}(x-a) + \ldots$, So $T''(a) = \frac{2!f^{(2)}(a)}{2!} + 0 + 0 + \cdots = f^{(2)}(a)$.
- $T^{(3)}(x) = 0 + 0 + 0 + \frac{3!f^{(3)}(a)}{3!} + \ldots etc\ldots$ So $T^{(3)}(a) = \frac{3!f^{(3)}(a)}{3!} + 0 + \cdots = f^{(3)}(a)$.
- etc.....

Example (McLaurin Series.)

Example Find the McLaurin Series of the function $f(x) = e^x$. Find the radius of convergence of this series.

- We need to calculate the derivatives of $f(x)$ and evaluate them at 0.
- $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x, \dots, f^{(n)}(x) = e^x$.
- $f(0) = e^0 = 1$, $f'(0) = e^0 = 1$, $f''(0) = e^0 = 1$, $\dots$ $f^{(n)}(0) = e^0 = 1$.
- The McLaurin series for $f(x) = e^x$ is given by $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \cdots$
- When we plug in the values for $f^n(0)$ from above, we get that the McL series for $f(x) = e^x$ is given by $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- Recall that last day we showed that this series converges for all values of x. We have yet to show that it converges to e^x.
- Because this series converges for all values of x, we have the following important limit:

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0 \quad \text{for all values of} \quad x.$$

Example (McLaurin Series)

Example Find the McLaurin Series of the function $f(x) = \sin x$. Find the radius of convergence of this series.

- We need to calculate the derivatives of $f(x)$ and evaluate them at 0.
- $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^4(x) = \sin x \dots$, $f^{(n)}(x) = \text{complicated}$.
- $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3)}(0) = -1$, $f^{(4)}(0) = 0 \dots$
  $$f^{(n)}(0) = \begin{cases} 0 & \text{if} \quad n \text{ is even} \\ \pm 1 & \text{if} \quad n \text{ is odd} \end{cases}$$
- The McLaurin series for $f(x) = \sin x$ is given by
  $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \cdots$
- When we plug in the values for $f^{(n)}(0)$ from above, we get that the McL series for $f(x) = \sin x$ is given by
  $0 + \frac{x}{1!} + 0 + \frac{(-1)x^3}{3!} + 0 + \frac{x^5}{5!} + 0 + \frac{(-1)x^7}{7!} \cdots$
- which we can write with summation notation as $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.
- To check the radius of convergence of this series, we use the ratio test, $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{2n+3}/(2n+3)!}{|x|^{2n+1}/(2n+1)!} = \lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0$ for all values of x.
- Therefore the radius of convergence is ∞.

Example (Taylor series expansion of e^x at 1)

Example Find the Taylor series expansion of the function $f(x) = e^x$ at $a = 1$. Find the radius of convergence of this series.

- We calculate the derivatives of $f(x)$ and evaluate them at 1.
- $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x, \dots$, $f^{(n)}(x) = e^x$.
- $f(1) = e^1 = e$, $f'(1) = e^1 = e$, $f''(1) = e^1 = e$, $\dots$ $f^{(n)}(1) = e^1 = e$.
- The Taylor series for $f(x) = e^x$ at $\boxed{a = 1}$ is given by $\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \cdots$
- When we plug in the values for $f^{(n)}(1)$ from above, we get that the Taylor series for $f(x) = e^x$ at $a = 1$ is given by $\sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!} = e + \frac{e(x-1)}{1!} + \frac{e(x-1)^2}{2!} + \frac{e(x-1)^3}{3!} + \cdots$
- To check the radius of convergence of this series, we use the ratio test, $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x-1|^{n+1}/(n+1)!}{|x-1|^n/(n)!} = \lim_{n\to\infty} \frac{|x-1|}{(n+1)} = 0$ for all values of x.
- Therefore the radius of convergence is ∞.
- In fact it can be shown that this series also converges to e^x everywhere. (F.Y.I. Even though the partial sums differ from the McL series of e^x, both series turn out to be the same.)

Answer to Q1

Theorem If f has a power series expansion at a, that is if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for all} \quad x \text{ such that } |x-a| < R$$

for some $R > 0$, then that power series is the Taylor series of f at a. We must have

$$c_n = \frac{f^{(n)}(a)}{n!} \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all x such that $|x-a| < R$.

If $a = 0$ the series in question is the McLaurin series of f.

- **Example** This result is saying that **if** $f(x) = e^x$ has a power series expansion at 0, then that power series expansion must be the McLaurin series of e^x which is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

However the result is **not saying** that e^x sums to this series. To prove that we need to use Taylor's theorem below.

Answer to question 1

- **Example** The result also says that IF $f(x) = e^x$ has a power series expansion at 1, then that power series expansion must be

$$e + e(x-1) + \frac{e(x-1)^2}{2!} + \frac{e(x-1)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!}$$

  However, we must use Taylor's theorem on the remainder to show that this series sums to $f(x) = e^x$ for all values of x.

- **Example** Also we have that IF $\sin x$ has a power series expansion at 0, then that power series expansion must be $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$.

Q2: When does $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$?

Our second question now becomes:
For which values of x does the Taylor series of f at a converge to $f(x)$?

For any value of x, the Taylor series of the function $f(x)$ about $x = a$ converges to $f(x)$ when the partial sums of the series ($T_n(x)$ below) converge to $f(x)$. We let

$$\boxed{R_n(x) = f(x) - T_n(x)},$$

where

$$T_n(x) = f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f^{(2)}(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$T_n(x)$ given above is called **the nth Taylor polynomial of f at a** and $R_n(x)$ is called the **remainder** of the Taylor series.

- **Theorem** Let $f(x)$, $T_n(x)$ and $R_n(x)$ be as above. If

$$\boxed{\lim_{n\to\infty} R_n(x) = 0 \quad \text{for} \quad |x-a| < R,}$$

then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Taylor's Theorem on the remainder

The following theorem is crucial in calculating $\lim_{n\to\infty} R_n(x)$ on an interval around a:

Taylor's Inequality If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$ then the remainder $R_n(x)$ of the Taylor Series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for} \quad |x-a| \leq d.$$

- **Example: Taylor's Inequality applied to $\sin x$.** If $f(x) = \sin x$, then for any n, $f^{(n+1)}(x)$ is either $\pm\sin x$ or $\pm\cos x$. In either case $|f^{(n+1)}(x)| \leq 1$ for all values of x. Therefore, with $M = 1$ and $a = 0$ and d any number, Taylor's inequality tells us that
$|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1} \quad \text{for} \quad |x| \leq d \quad (= \infty \text{ here}).$
- **Example: Taylor's Inequality applied to e^x.** If $h(x) = e^x$, then for any value of n, $h^{(n+1)}(x) = e^x$. Now if d is any number, I know that $|h^{(n+1)}(x)| = |e^x| < e^d$ for all x with $|x| < d$. Hence applying Taylor's inequality to the McLaurin series for e^x (with $a = 0$) we get that
$|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{n+1} \quad \text{for} \quad |x| \leq d.$

Answers to Question 2

Example Prove that $\sin x$ is equal to the sum of its McLaurin series for all x, that is, show that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

for all x.

- I need to show that for any value of x, the remainder $R_n(x) = \sin(x) - T_n(x)$ has the property that $\lim_{n\to\infty} |R_n(x)| = 0$.
- When we apply Taylor's theorem to the remainder (as shown above), we get $|f^{(n+1)}(x)| \leq 1$ and $|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}$ for all x.
- Therefore $0 \leq \lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{1}{(n+1)!}|x|^{n+1} = 0$ for all x
- Therefore

  $$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

  for all x.

Answers to Question 2

Example Prove that e^x is equal to the sum of its McLaurin series for all x, that is, show that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

for all x.

- I need to show that for any value of x, the remainder $R_n(x) = e^x - T_n(x)$ has the property that $\lim_{n\to\infty} |R_n(x)| = 0$.
- When we apply Taylor's theorem to the remainder (as shown above), we get $|f^{(n+1)}(x)| \le e^d$ and $|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1}$ for all x with $|x| < d$, where d can be chosen arbitrarily.
- Therefore $0 \le \lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = 0$ for all x with $|x| < d$.
- Therefore $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots$ for all x with $|x| < d$.
- Since d can be chosen to be as big as I like, I can conclude that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots \quad \text{for all } x$$

Power series expansion of $\cos x$.

Example Find a power series representation for $\cos x$.

- We have $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
- Since $\frac{d \sin x}{dx} = \cos x$, we can differentiate both sides of the above equation to get

$$\cos x = \sum_{n=0}^{\infty} \frac{d(-1)^n \frac{x^{2n+1}}{(2n+1)!}}{dx} = \frac{dx}{dx} - \frac{d\frac{x^3}{3!}}{dx} + \frac{d\frac{x^5}{5!}}{dx} \cdots$$

- Therefore

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} \cdots$$

- So

$$\boxed{\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots}$$

Apps (Summing series)

We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

for all x.

- Therefore
$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$
- and
$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!} = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \cdots$$
- and
$$\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Apps (Finding Limits)

Example use power series to find the limit

$$\lim_{x \to 0} \frac{\cos(x^5) - 1}{x^{10}}$$

(This is a long computation if you use L'Hopital's rule).

- We have
$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$
- and hence by substitution, we have
$$\cos(x^5) = 1 - \frac{x^{10}}{2!} + \frac{x^{20}}{4!} - \frac{x^{30}}{6!} \cdots$$
- Therefore $\cos(x^5) - 1 = -\frac{x^{10}}{2!} + \frac{x^{20}}{4!} - \frac{x^{30}}{6!} \cdots$
- and $\frac{\cos(x^5)-1}{x^{10}} = -\frac{1}{2} + \frac{x^{10}}{4!} - \frac{x^{20}}{6!} \cdots$
- Since power series (with real x values) are continuous functions we have $\lim_{x \to 0} \frac{\cos(x^5)-1}{x^{10}} = \frac{-1}{2}$, which is the value of the power series on the RHS when $x = 0$.

Well Known Power Series expansions

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$