Lecture 30: Applications of Taylor Series

Recall that we used the linear approximation of a function in Calculus 1 to estimate the values of the function near a point a (assuming f was differentiable at a):

$$\boxed{f(x) \approx f(a) + f'(a)(x-a) \quad \text{for} \quad x \quad \text{near} \quad a.}$$

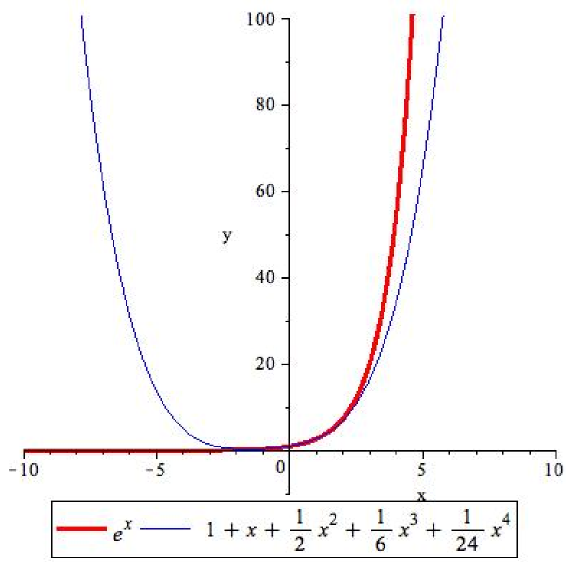
Now suppose that $f(x)$ has infinitely many derivatives at a and $f(x)$ equals the sum of its Taylor series in an interval around a, then we can approximate the values of the function $f(x)$ near a by the nth partial sum of the Taylor series at x, or the nth Taylor Polynomial:

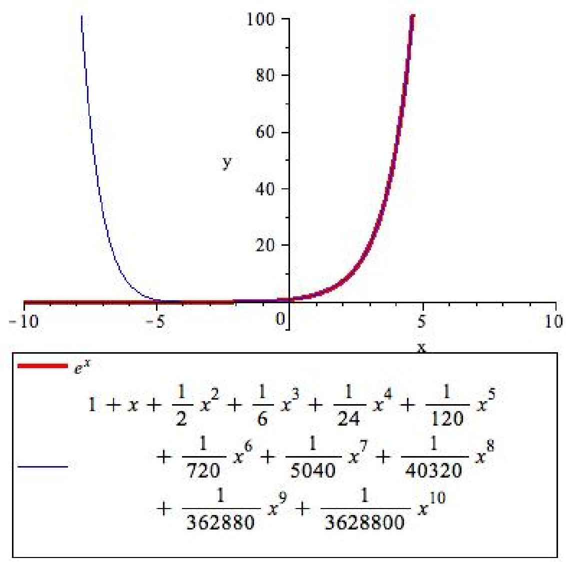
$$\boxed{f(x) \approx T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.}$$

$T_n(x)$ is a polynomial of degree n with the property that $T_n(a) = f(a)$ and $T_n^{(i)}(a) = f^{(i)}(a)$ for $i = 1, 2, \ldots, n$.

Note that $T_1(x)$ is the linear approximation given above.

Example For example, we could estimate the values of $f(x) = e^x$ on the interval $-4 < x < 4$, by either the fourth degree Taylor polynomial at 0 or the tenth degree Taylor. The graphs of both are shown below.





If $f(x)$ equals the sum of its Taylor series (about a) at x, then we have

$$\lim_{n\to\infty} T_n(x) = f(x)$$

and larger values of n will give better approximations to $f(x)$. We can use Taylor's Inequality to help estimate the error in our approximation.

The error in our approximation of $f(x)$ by $T_n(x)$ is $|R_n(x)| = |f(x) - T_n(x)|$. We can estimate the size of this error in two ways:

1. Taylor's Inequality If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$ then the remainder $R_n(x)$ of the Taylor Series satisfies the inequality

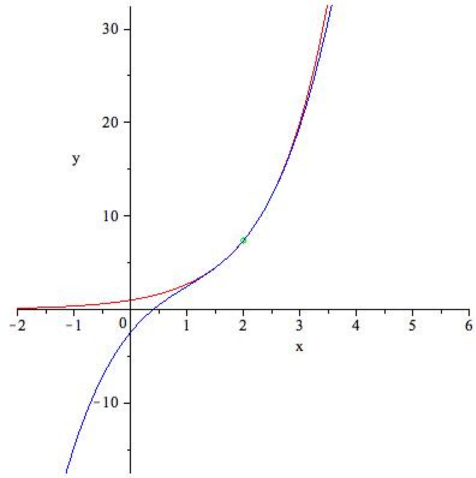
$$\boxed{|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for} \quad |x-a| \leq d.}$$

2. If the Taylor series is an alternating series, we can use the alternating series estimate for the error.

Example (a) Consider the approximation to the function $f(x) = e^x$ by the fourth MacLaurin polynomial of $f(x)$ given above. How accurate is the approximation when $-4 \leq x \leq 4$? (Give an upper bound for the error on this interval).

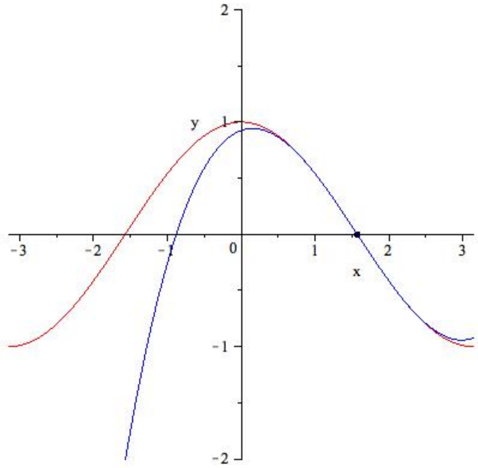
(b) Find an interval around 0 for which this approximation has an error less than .001.

Example (a) Find the third Taylor polynomial of $f(x) = e^x$ at $a = 2$.



(b) Use Taylor's Inequality to give an upper bound for the error possible in using this approximation to e^x for $1 < x < 3$.

Example (a) Find the third Taylor polynomial of $g(x) = \cos x$ at $a = \frac{\pi}{2}$.



(b) Use the fact that the Taylor series is an alternating series to determine the maximum error possible in using this approximation to $\cos x$ for $\frac{\pi}{4} \le x \le \frac{3\pi}{4}$.