Applications of Taylor Series

Recall that we used the linear approximation of a function in Calculus 1 to estimate the values of the function near a point a (assuming f was differentiable at a):

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{for} \quad x \quad \text{near} \quad a.$$

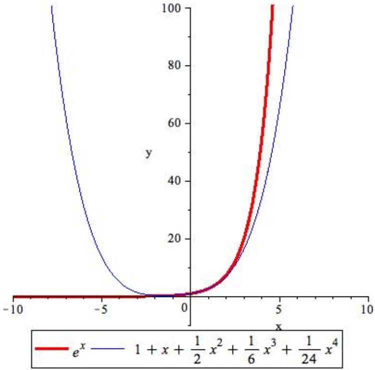
Now suppose that $f(x)$ has infinitely many derivatives at a and $f(x)$ equals the sum of its Taylor series in an interval around a, then we can approximate the values of the function $f(x)$ near a by the nth partial sum of the Taylor series at x, or the nth Taylor Polynomial:

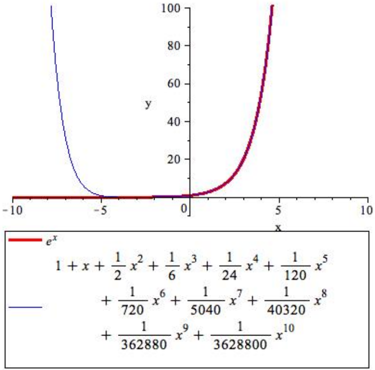
$$f(x) \approx T_n(x) \\ = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$T_n(x)$ is a polynomial of degree n with the property that $T_n(a) = f(a)$ and $T_n^{(i)}(a) = f^{(i)}(a)$ for $i = 1, 2, \ldots, n$.
Note that $T_1(x)$ is the linear approximation given above.

Example

Example For example, we could estimate the values of $f(x) = e^x$ on the interval $-4 < x < 4$, by either the fourth degree Taylor polynomial at 0 or the tenth degree Taylor. The graphs of both are shown below.





Approximations

If $f(x)$ equals the sum of its Taylor series (about a) at x, then we have

$$\lim_{n\to\infty} T_n(x) = f(x)$$

and larger values of n should give of better approximations to $f(x)$. The approximation We can use Taylor's Inequality to help estimate the error in our approximation.

The error in our approximation of $f(x)$ by $T_n(x)$ is $|R_n(x)| = |f(x) - T_n(x)|$. We can estimate the size of this error in two ways:

- **1. Taylor's Inequality** If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$ then the remainder $R_n(x)$ of the Taylor Series satisfies the inequality

$$\boxed{|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for} \quad |x-a| \leq d.}$$

- **2.** If the Taylor series is an alternating series, we can use the alternating series estimate for the error.

Example

Example (a) Consider the approximation to the function $f(x) = e^x$ by the fourth McLaurin polynomial of $f(x)$ given above.

- $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$.

(b) How accurate is the approximation when $-4 \leq x \leq 4$? (Give an upper bound for the error on this interval).

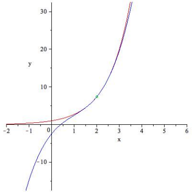
- We have $\frac{d^5 e^x}{dx^5} = e^x$ and hence $\left|\frac{d^5 e^x}{dx^5}\right| < e^4$ if $|x| < 4$.
- If $|x| < 4$, Taylor's inequality says that $|R_n(x)| \leq \frac{e^4}{(5)!}|x|^5 < \frac{e^4}{(5)!}|4|^5 = 465.9$ on this interval.
- This is a conservative estimate of the error on this interval. In fact $|R_n(x)| \leq 65$ on this interval.

(c) Find an interval around 0 for which this approximation has error $< .001$.

- By Taylor's approximation, If x is in the interval $(-r, r)$, then $|R_n(x)| \leq \frac{e^r}{(5)!}|x|^5 \leq \frac{e^r}{(5)!}|r|^5$.
- To find such an r with $|R_n(x)| \leq .001$, it suffices to find a value of r for which $\frac{e^r}{(5)!}|r|^5 \leq .001$.
- If we assume that $r < 1$, we have $e^r < e$ and we need an r with $\frac{e}{(5)!}|r|^5 \leq .001$ or $|r|^5 < \frac{.001 \times 5!}{e}$. This works if $r < \sqrt[5]{\frac{.001 \times 5!}{e}} \approx 0.53$

Example: Estimating values of e^x,

Example (a) Find the third Taylor polynomial of $f(x) = e^x$ at $a = 2$.



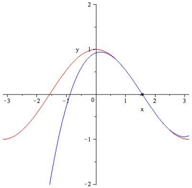
- $T_3(x) = f(2) + f'(2)(x-2) + \frac{f^{(2)}(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3$
- $= e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^3}{3!}(x-2)^3$.

(b) Use Taylor's Inequality to give an upper bound for the error possible in using this approximation to e^x for $1 < x < 3$.

- By Taylor's theorem, we have $|R_n(x)| = |e^x - T_3(x)| \leq \frac{M|x-2|^4}{4!}$, where $M =$ max $|f^{(4)}(x)|$ on the interval $(1, 3)$.
- $M = e^3$ works and hence the error of approximation $= |R_n(x)| \leq \frac{e^3|x-2|^4}{4!} \leq \frac{e^3}{4!} = .837$ for any x in $(1, 3)$.

Example

Example (a) Find the third Taylor polynomial of $g(x) = \cos x$ at $a = \frac{\pi}{2}$.



- $g(x) = \cos x, g'(x) = -\sin x,$
  $g''(x) = -\cos x, g^{(3)}(x) = \sin x.$
- $g(\frac{\pi}{2}) = 0, \quad g'(\frac{\pi}{2}) = -1,$
  $g''(\frac{\pi}{2}) = 0, \quad g^{(3)}(\frac{\pi}{2}) = 1.$
- $T_3(x) = g(\frac{\pi}{2}) + g'(\frac{\pi}{2})(x - \frac{\pi}{2}) + \frac{g''(\frac{\pi}{2})}{2!}(x - \frac{\pi}{2})^2 + \frac{g^{(3)}(\frac{\pi}{2})}{3!}(x - \frac{\pi}{2})^3$
- $T_3(x) = -(x - \frac{\pi}{2}) + \frac{(x - \frac{\pi}{2})^3}{3!}.$

(b) Use the fact that the Taylor series is an alternating series to determine the maximum error possible in using this approximation to $\cos x$ for $\frac{\pi}{4} \le x \le \frac{3\pi}{4}$?

- At any point x in $(\frac{\pi}{4}, \frac{3\pi}{4})$ the Taylor series for $\cos x$ at $a = \frac{\pi}{2}$ is an alternating series converging to $\cos x$:
  $T(x) = -(x - \frac{\pi}{2}) + \frac{(x - \frac{\pi}{2})^3}{3!} - \frac{(x - \frac{\pi}{2})^5}{5!} \dots$
- Therefore the error from the above approximation is
  $|R_n(x)| = |\cos x - T_3(x)| \le \left| \frac{(x - \frac{\pi}{2})^5}{5!} \right| \le \frac{\left(\frac{\pi}{4}\right)^5}{5!} = \frac{\pi^5}{4^5 5!} = .0024.$