Curves defined by Parametric equations

When the path of a particle moving in the plane is not the graph of a function, we cannot describe it using a formula that express y directly in terms of x, or x directly in terms of y. Instead, we need to use a third variable t, called a **parameter** and write:

$$x = f(t) \qquad y = g(t)$$

- The set of points $(x, y) = (f(t), g(t))$ described by these equations when t varies in an interval I form a curve, called a **parametric curve**, and $x = f(t), y = g(t)$ are called the **parametric equations** of the curve. Often, t represents time and therefore we can think of $(x, y) = (f(t), g(t))$ as the position of a particle at time t.
- If I is a closed interval, $a \leq t \leq b$, the point $(f(a), g(a))$ is the **initial point** and the point $(f(b), g(b))$ is the **terminal point**.

Example 1

Example 1 Draw and identify the parametric curve given by the parametric equations:

$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$$

t	x	y
0		
$\frac{\pi}{4}$		
$\frac{\pi}{2}$		
$\frac{3\pi}{4}$		
π		
$\frac{3\pi}{2}$		
2π		

- First we look at points on the curve for particular values of t (in table).

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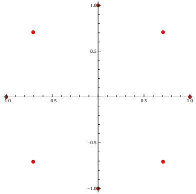
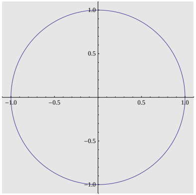
π	-1	0
$\frac{3\pi}{2}$	0	-1
2π	1	0

- First we look at points on the curve for particular values of t (in table).
- Plotting these points, we get points on a circle.





- Filling in the details, we see that if as t increases from 0 to 2π, the points trace out a circle of radius 1 in an anti-clockwise direction, with I.P. $(1, 0)$ and T.P. $(1, 0)$.

Example 2

As t increases, we travel along the curve in a particular direction giving the curve an **orientation** which is often indicated by arrows. The curve however may pass through a section of the curve repeatedly and may do so in different directions for different values of t.

Example 2 Describe the parametric curve represented by the parametric equations: $x = \sin 2t \qquad y = \cos 2t \qquad 0 \leq t \leq 2\pi$

t	x	y
0		
$\pi/4$		
$\pi/2$		
$3\pi/4$		
π		
$3\pi/2$		
2π		

- We find some points on the curve (see table)

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$\pi/2$		
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$\pi/4$	1	0
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$3\pi/4$	-1	0
π		
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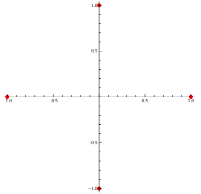
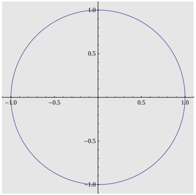
t	x	y
$(I.P.)$ 0	0	1
$\pi/4$	1	0
$\pi/2$	0	-1
$3\pi/4$	-1	0
π	0	1
$3\pi/2$	0	-1
$(T.P.)$ 2π	0	1

- We find some points on the curve (see table)
- We see that the points are on the unit circle, but the curve sweeps around it twice in a clockwise direction.





Example 2

Note The curve in examples 1 and 2 are the same but the parametric curve are not. Because in one case the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from $(1, 0)$. In example 2 instead, the point $(x, y) = (\sin 2t, \cos 2t)$ moves twice around the circle in the clockwise direction starting from $(0, 1)$.

Example 3

Example 3 Sketch the graph of the curve described by the following set of parametric equations.

$$x = t^3 - t, \quad y = t^2, \quad 0 \leq t < \infty$$

t	x	y
0		
0.5		
1		
1.5		
2		

- We find some points on the curve (see table)

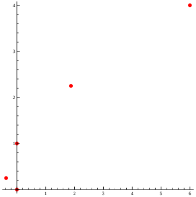
Example 3

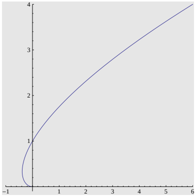
Example 3 Sketch the graph of the curve described by the following set of parametric equations.

$$x = t^3 - t, \quad y = t^2, \quad 0 \leq t < \infty$$

t	x	y
0	0	0
0.5	-0.375	0.25
1	0	1
1.5	1.875	2.25
2	6	4

- We find some points on the curve (see table)
- We plot the points and join them to get a curve similar to the one shown.





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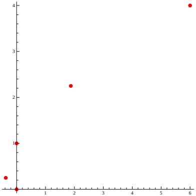
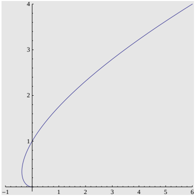
- So far, we have the following picture:

  What happens after $t = 2$? Can the curve turn around again?

- I claim that $x(t)$ and $y(t)$ are always increasing for $t \geq 2$ and hence the curve cannot turn again.
- We have $x'(t) = 3t^2 - 1$, therefore $x'(t) = 0$ when $t = \pm\frac{1}{\sqrt{3}}$. Since $x'(1) = 2 > 0$, we can conclude that $x'(t) > 0$ on the interval $(\frac{1}{\sqrt{3}}, \infty)$ and therefore, the values of x are increasing on that interval.
- Since $y'(t) = 2t > 0$ for $t > 2$, we see that the values of y are increasing when $t > 2$.
- Therefore there is only one turning point on the curve.

Converting: Parametric to Cartesian

There is no exact method for converting parametric equations for a curve to an equation in x and y only. If we can solve for t in terms of either x or y, we can substitute this for the value of t in one of the equations to get an equation in x and y only.

Example 4 Covert the following parametric equation to an equation relating x and y:

$$x = 2t + 1, \quad y = t - 2, \quad -\infty < t < \infty$$

- We can solve for t in terms of y. Since $y = t - 2$, we have $t = y + 2$.
- Substituting this expression for t in the equation for x, we get $x = 2(y + 2) + 1 = 2y + 5$.
- Therefore all points on the parametric curve are on the line $x = 2y + 5$. Since t takes all values in the interval $(-\infty, \infty)$, y also runs through all values from $-\infty$ to ∞ and the parametric curve describes the entire line.

Example: Parametric to Cartesian

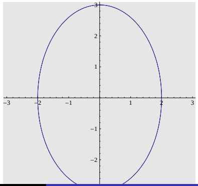
Sometimes, we can see a relationship between the x and y co-ordinates and thus eliminate the t.

Example 5 Convert the following parametric equation to an equation relating x and y:

$$x = 2\cos t \qquad y = 3\sin t$$

and describe the curve traced when $0 \leq t \leq 4\pi$.

- We know that $\cos^2 t + \sin^2 t = 1$.
- Therefore
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$
- The curve traced by this parametric equation is therefore an ellipse. If we check some points we see that the initial point is at $(2, 0)$ and the ellipse is traced in an anti-clockwise direction. Because the point on the curve at $t + 2\pi$ is the same as the point at t, we see that the curve is traced twice.



Converting: Cartesian to Parametric

Easy cases If a curve is defined by the equation $y = f(x)$, the equations $x = t$ and $y = f(t)$ give parametric equations describing the curve.
If a curve is described by the equation $x = g(y)$, the equations $x = t$ and $x = g(t)$ give parametric equations describing the curve.
Example 6 Give parametric equations describing the graph of the parabola $y = x^2$.

- We can let $x = t$, then $y = t^2$.
- We trace out the entire parabola from left to right if we let t run from $-\infty$ to ∞.

Example: Cartesian to Parametric

Example 7 Find parametric equations on $0 \leq t \leq 2\pi$ for the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$ twice counterclockwise.

- Each point has the form $x(t) = a\cos 2t$, $y(t) = a\sin 2t$, where $2t$ is the size of the angle that a ray from the point to the origin makes with the positive x axis.
- Since $0 \leq t \leq 2\pi$ implies that $0 \leq 2t \leq 4\pi$, we get that the particle sweeps around the circle twice.

Extra Example: Parametric to Cartesian

Example 5 Convert the following parametric equation to an equation relating x and y:

$$x = \sin t \qquad y = \cos^2 t$$

and describe the curve traced when $0 \le t \le 2\pi$.

- We know that $\cos^2 t + \sin^2 t = 1$.
- Therefore
$$x = \sin t \qquad y = \cos^2 t = 1 - x^2$$
- The curve traced by this parametric equation is therefore on the graph of $y = 1 - x^2$.
- I.P. $(0, 1)$, $t = \frac{\pi}{2} \quad \to \quad (1, 0)$, $\quad t = \pi, \quad \to \quad (0, 1)$, $\quad t = \frac{3\pi}{2} \to \quad (-1, 0)$, $t = 2\pi \to \quad (0, 1)$.
- We see that the particle oscillates back and forth between $(1, 0)$ and $(-1, 0)$.