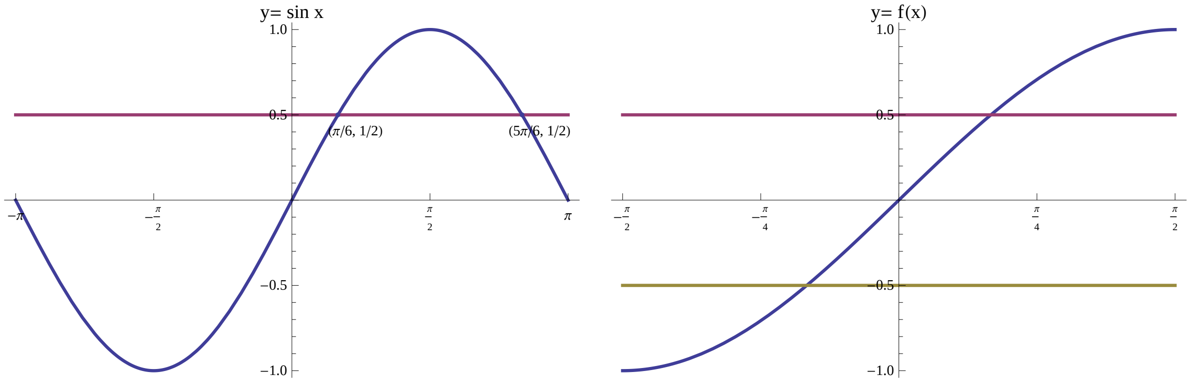
Lecture 6 : Inverse Trigonometric Functions

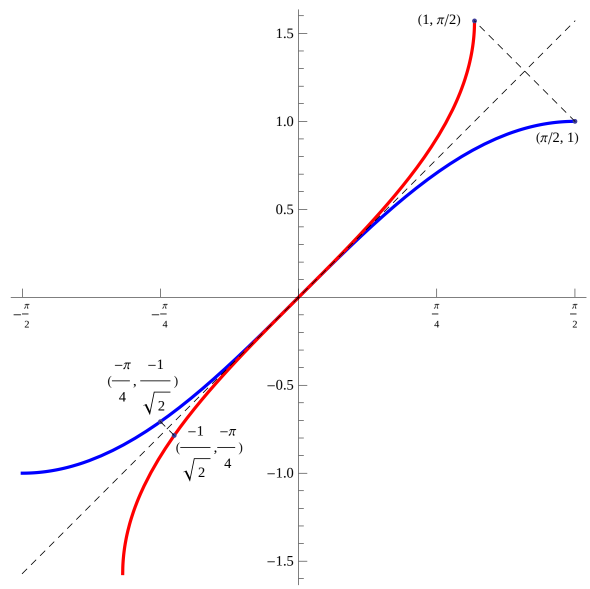
Inverse Sine Function (arcsin x = $sin^{-1}x$**)** The trigonometric function $\sin x$ is not one-to-one functions, hence in order to create an inverse, we must restrict its domain.
The restricted sine function is given by

$$f(x) = \begin{cases} \sin x & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \text{undefined} & \text{otherwise} \end{cases}$$

We have Domain(f) $= [-\frac{\pi}{2}, \frac{\pi}{2}]$ and Range(f) $= [-1, 1]$.



We see from the graph of the restricted sine function (or from its derivative) that the function is one-to-one and hence has an inverse, shown in red in the diagram below.



This inverse function, $f^{-1}(x)$, is denoted by

$$\boxed{f^{-1}(x) = \sin^{-1} x \quad \text{or} \quad \arcsin x.}$$

Properties of $\sin^{-1} x$**.**

Domain($\sin^{-1}$) $= [-1, 1]$ and Range($\sin^{-1}$) $= [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Since $f^{-1}(x) = y$ if and only if $f(y) = x$, we have:

$$\boxed{\sin^{-1} x = y \text{ if and only if } \sin(y) = x \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}.}$$

Since $f(f^{-1})(x) = x \quad f^{-1}(f(x)) = x$ we have:

$$\boxed{\sin(\sin^{-1}(x)) = x \text{ for } x \in [-1, 1] \quad \sin^{-1}(\sin(x)) = x \text{ for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].}$$

from the graph: $\sin^{-1} x$ is an odd function and $\sin^{-1}(-x) = -\sin^{-1} x$.

Example Evaluate $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ using the graph above.

Example Evaluate $\sin^{-1}(\sqrt{3}/2)$, $\sin^{-1}(-\sqrt{3}/2)$,

Example Evaluate $\sin^{-1}(\sin \pi)$.

Example Evaluate $\cos(\sin^{-1}(\sqrt{3}/2))$.

Example Give a formula in terms of x for $\tan(\sin^{-1}(x))$

Derivative of $\sin^{-1} x$.

$$\boxed{\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}, \quad -1 \le x \le 1.}$$

Proof We have $\sin^{-1} x = y$ if and only if $\sin y = x$. Using implicit differentiation, we get $\cos y \frac{dy}{dx} = 1$ or

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

Now we know that $\cos^2 y + \sin^2 y = 1$, hence we have that $\cos^2 y + x^2 = 1$ and

$$\cos y = \sqrt{1 - x^2}$$

and

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}.$$

If we use the chain rule in conjunction with the above derivative, we get

$$\boxed{\frac{d}{dx}\sin^{-1}(k(x)) = \frac{k'(x)}{\sqrt{1-(k(x))^2}}, \quad x \in \text{Dom}(k) \text{ and } -1 \leq k(x) \leq 1.}$$

Example Find the derivative

$$\frac{d}{dx}\sin^{-1}\sqrt{\cos x}$$

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Inverse Cosine Function We can define the function $\cos^{-1}x = \arccos(x)$ similarly. The details are given at the end of this lecture.

$$\text{Domain}(\cos^{-1}) = [-1,1] \quad \text{and} \quad \text{Range}(\cos^{-1}) = [0,\pi].$$

$$\boxed{\cos^{-1}x = y \quad \text{if and only if} \quad \cos(y) = x \text{ and } 0 \leq y \leq \pi.}$$

$$\boxed{\cos(\cos^{-1}(x)) = x \text{ for } x \in [-1,1] \quad \cos^{-1}(\cos(x)) = x \text{ for } x \in \big[0,\pi\big].}$$

It is shown at the end of the lecture that

$$\frac{d}{dx}\cos^{-1}x = -\frac{d}{dx}\sin^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

and one can use this to prove that

$$\boxed{\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}}.$$

Inverse Tangent Function

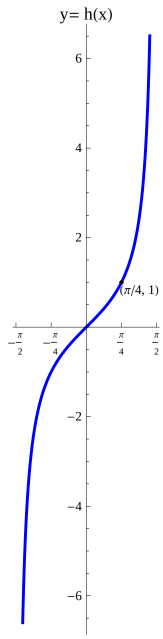
The tangent function is not a one to one function, however we can also restrict the domain to construct a one to one function in this case.

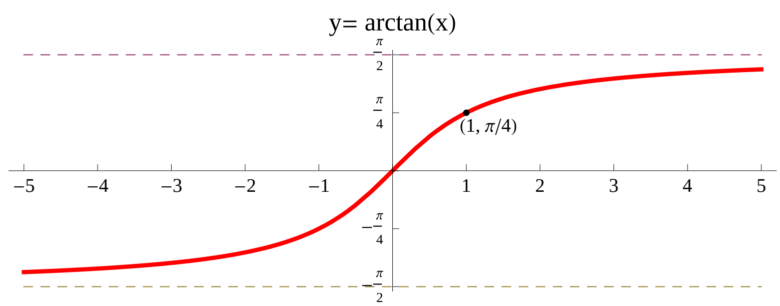
The restricted tangent function is given by

$$h(x) = \begin{cases} \tan x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \text{undefined} & \text{otherwise} \end{cases}$$

We see from the graph of the restricted tangent function (or from its derivative) that the function is one-to-one and hence has an inverse, which we denote by

$$\boxed{h^{-1}(x) = \tan^{-1}x \quad \text{or} \quad \arctan x.}$$





Properties of $tan^{-1}x$.

Domain$(\tan^{-1}) = (-\infty, \infty)$ and Range$(\tan^{-1}) = (-\frac{\pi}{2}, \frac{\pi}{2})$.

Since $h^{-1}(x) = y$ if and only if $h(y) = x$, we have:

$$\boxed{\tan^{-1} x = y \text{ if and only if } \tan(y) = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}.}$$

Since $h(h^{-1}(x)) = x$ and $h^{-1}(h(x)) = x$, we have:

$$\boxed{\tan(\tan^{-1}(x)) = x \text{ for } x \in (-\infty, \infty) \quad \tan^{-1}(\tan(x)) = x \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).}$$

Frpm the graph, we have: $\boxed{\tan^{-1}(-x) = -\tan^{-1}(x).}$

Also, since $\lim_{x \to (\frac{\pi}{2}^-)} \tan x = \infty$ and $\lim_{x \to (-\frac{\pi}{2}^+)} \tan x = -\infty$,

we have $\boxed{\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}}$ and $\boxed{\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}}$

Example Find $\tan^{-1}(1)$ and $\tan^{-1}(\frac{1}{\sqrt{3}})$.

Example Find $\cos(\tan^{-1}(\frac{1}{\sqrt{3}}))$.

Derivative of $\tan^{-1} x$.

$$\boxed{\frac{d}{dx} \tan^{-1} x = \frac{1}{x^2 + 1}, \quad -\infty < x < \infty.}$$

Proof We have $\tan^{-1} x = y$ if and only if $\tan y = x$. Using implicit differentiation, we get $\sec^2 y \frac{dy}{dx} = 1$ or

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y.$$

Now we know that $\cos^2 y = \cos^2(\tan^{-1} x) = \frac{1}{1+x^2}$. proving the result.

If we use the chain rule in conjunction with the above derivative, we get

$$\boxed{\frac{d}{dx}\tan^{-1}(k(x)) = \frac{k'(x)}{1+(k(x))^2}, \quad x \in \mathrm{Dom}(k)}$$

Example Find the domain and derivative of $\tan^{-1}(\ln x)$

Domain $= (0, \infty)$

$$\frac{d}{dx}\tan^{-1}(\ln x) = \frac{\frac{1}{x}}{1+(\ln x)^2} = \frac{1}{x(1+(\ln x)^2)}$$

Integration formulas

Reversing the derivative formulas above, we get

$$\boxed{\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C, \qquad \int \frac{1}{x^2+1}\, dx = \tan^{-1} x + C,}$$

Example

$$\int \frac{1}{\sqrt{9-x^2}}\, dx =$$

$$\int \frac{1}{3\sqrt{1-\frac{x^2}{9}}}\, dx = \int \frac{1}{3\sqrt{1-\frac{x^2}{9}}} dx = \frac{1}{3}\int \frac{1}{\sqrt{1-\frac{x^2}{9}}} dx$$

Let $u = \frac{x}{3}$, then $dx = 3du$ and

$$\int \frac{1}{\sqrt{9-x^2}}\, dx = \frac{1}{3}\int \frac{3}{\sqrt{1-u^2}} du = \sin^{-1} u + C = \sin^{-1}\frac{x}{3} + C$$

Example

$$\int_0^{1/2} \frac{1}{1+4x^2}\, dx$$

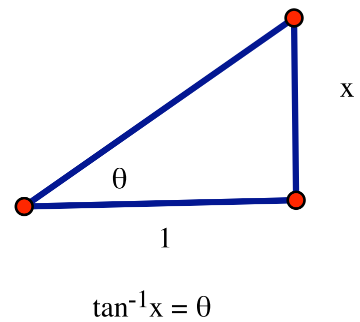
Let $u = 2x$, then $du = 2dx$, $\quad u(0) = 0, \quad u(1/2) = 1$ and

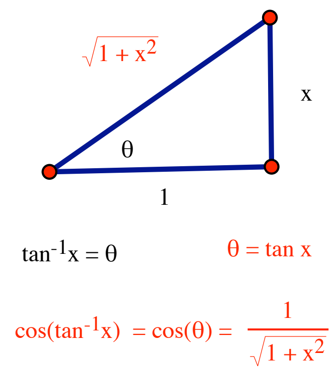
$$\int_0^{1/2} \frac{1}{1+4x^2}\, dx = \frac{1}{2}\int_0^1 \frac{1}{1+u^2}\, dx = \frac{1}{2}\tan^{-1} u|_0^1 = \frac{1}{2}[\tan^{-1}(1) - \tan^{-1}(0)]$$

$$\frac{1}{2}[\frac{\pi}{4} - 0] = \frac{\pi}{8}.$$

Extra Examples

Example Find a formula in terms of x for $\cos(\tan^{-1} x)$.





Example

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}}\, dx$$

Let $u = x^2$. Then $du = 2xdx$ and $\frac{du}{2} = xdx$. $u(0) = 0$ and $u(\frac{1}{\sqrt{2}}) = \frac{1}{2}$. We get

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}}\, dx = \frac{1}{2}\int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}(u)|_0^{\frac{1}{2}}$$

$$= \frac{1}{2}[\sin^{-1}(1/2) - \sin^{-1}(0)] = \frac{1}{2}[\pi/6 - 0] = \pi/12$$

Example

$$\int \frac{1}{x(1+(\ln x)^2)}\, dx$$

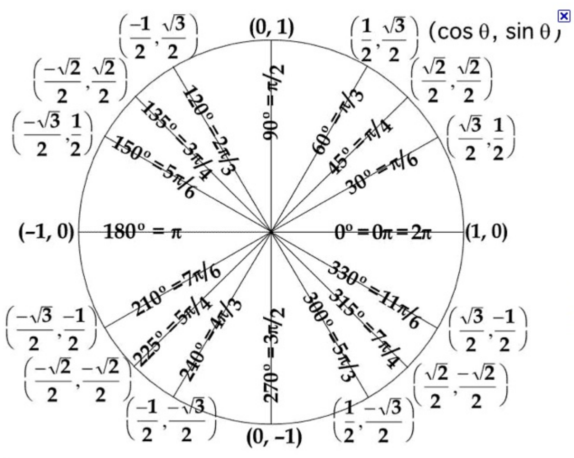
Let $u = \ln x$, then $du = dx/x$ and

$$\int \frac{1}{x(1+(\ln x)^2)}\, dx = \int \frac{1}{(1+(u)^2)}\, du = \tan^{-1} u + C = \tan^{-1}(\ln x) + C$$

Example Find the domain and derivative of $h(x) = \sin^{-1}(x^2 - 1)$

The domain of this function is all values of x for which $x^2 - 1$ is in the domain of $\sin^{-1}$ which is $\{x| -1 \le x \le 1\}$. Therefore the domain of $h = \{x$such that $\; -1 \le x^2 - 1 \le 1\}$. Now $-1 \le x^2 - 1 \le 1$ is the same as $0 \le x^2 \le 2$, or $-\sqrt{2} \le x \le \sqrt{2}$. Therefore the domain of h is the interval $-\sqrt{2} \le x \le \sqrt{2}$.

$$\frac{d}{dx}\sin^{-1}(x^2-1) = \frac{1}{\sqrt{1-(x^2-1)^2}} \cdot 2x = \frac{2x}{\sqrt{1-(x^2-1)^2}}.$$

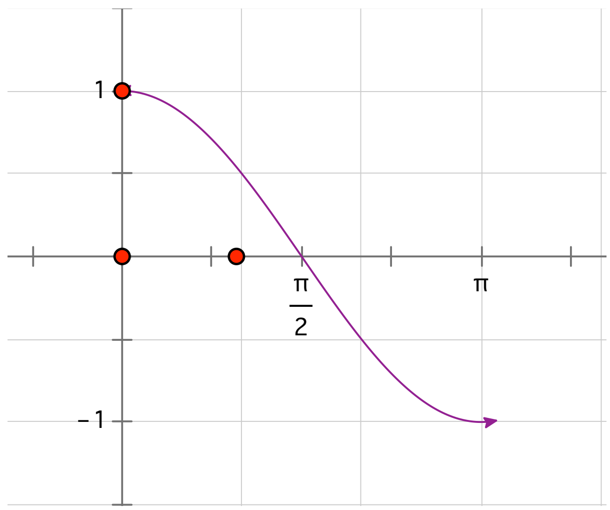


	0°	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

The restricted cosine function is given by

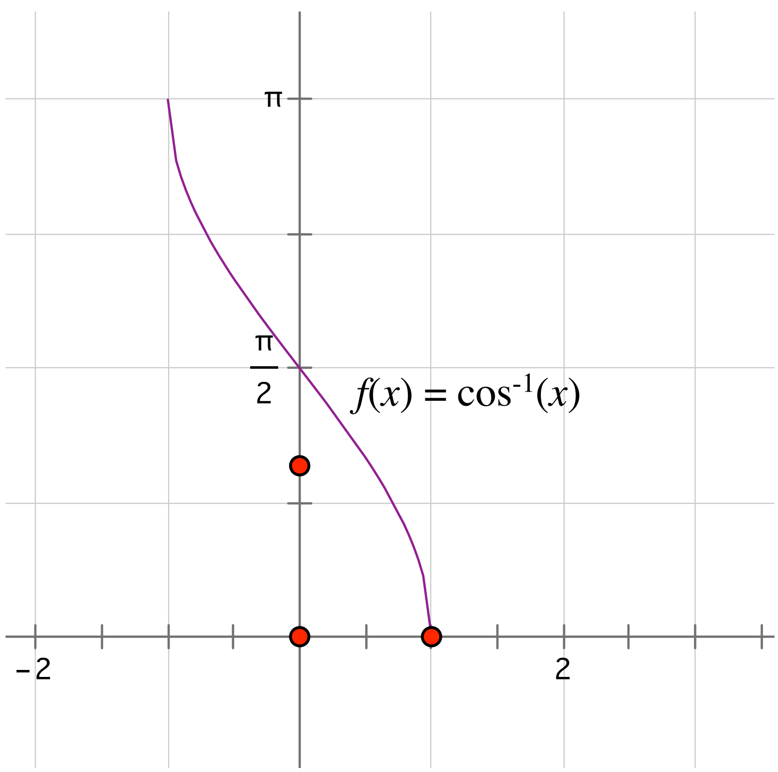
$$g(x) = \begin{cases} \cos x & 0 \le x \le \pi \\ \text{undefined} & \text{otherwise} \end{cases}$$

We have Domain(g) = $[0, \pi]$ and Range(g) = $[-1, 1]$.



We see from the graph of the restricted cosine function (or from its derivative) that the function is one-to-one and hence has an inverse,

$$\boxed{g^{-1}(x) = \cos^{-1} x \quad \text{or} \quad \arccos x}$$



$$\text{Domain}(\cos^{-1}) = [-1, 1] \quad \text{and} \quad \text{Range}(\cos^{-1}) = [0, \pi].$$

Recall from the definition of inverse functions:

$$g^{-1}(x) = y \ \text{ if and only if } \ g(y) = x.$$

$$\boxed{\cos^{-1} x = y \quad \text{if and only if} \quad \cos(y) = x \ \text{ and } \ 0 \le y \le \pi.}$$

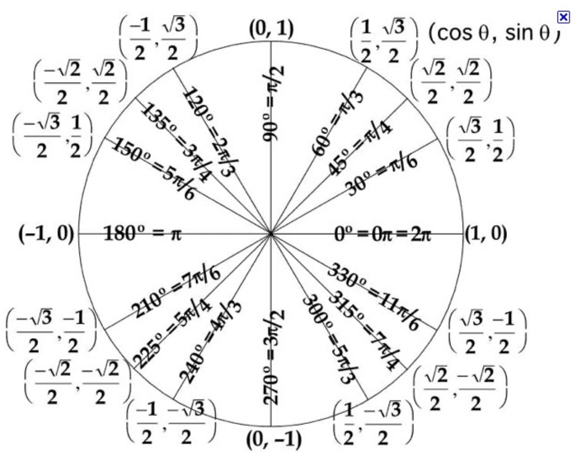
$$g(g^{-1}(x)) = x \quad g^{-1}(g(x)) = x$$

$$\boxed{\cos(\cos^{-1}(x)) = x \text{ for } \ x \in [-1, 1] \quad \cos^{-1}(\cos(x)) = x \ \text{ for } \ x \in [0, \pi].}$$

Note from the graph that $\boxed{\cos^{-1}(-x) = \pi - \cos^{-1}(x)}$.

$\cos^{-1}(\sqrt{3}/2) =$ ______ and $\cos^{-1}(-\sqrt{3}/2) =$ ______

You can use either chart below to find the correct angle between 0 and π.:

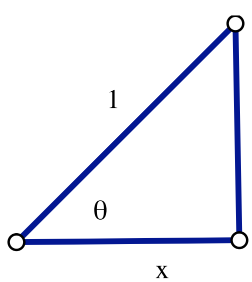


	$0°$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

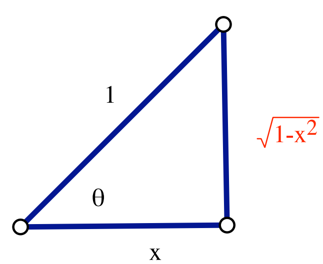
$\tan(\cos^{-1}(\sqrt{3}/2)) =$ ______

$\tan(\cos^{-1}(x)) =$ ______

Must draw a triangle with correct proportions:



$\cos\theta = x$



$\cos\theta = x$ $\qquad$ $\cos^{-1} x = \theta$

$$\tan(\cos^{-1} x) = \tan\theta = \frac{\sqrt{1-x^2}}{x}$$

$$\boxed{\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}, \quad -1 \le x \le 1.}$$

Proof We have $\cos^{-1}x = y$ if and only if $\cos y = x$. Using implicit differentiation, we get $-\sin y\frac{dy}{dx} = 1$ or

$$\frac{dy}{dx} = \frac{-1}{\sin y}.$$

Now we know that $\cos^2 y + \sin^2 y = 1$, hence we have that $\sin^2 y + x^2 = 1$ and

$$\sin y = \sqrt{1-x^2}$$

and

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}.$$

Note that $\frac{d}{dx}\cos^{-1}x = -\frac{d}{dx}\sin^{-1}x$. In fact we can use this to prove that $\boxed{\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}}$.

If we use the chain rule in conjunction with the above derivative, we get

$$\boxed{\frac{d}{dx}\cos^{-1}(k(x)) = \frac{-k'(x)}{\sqrt{1-(k(x))^2}}, \quad x \in \text{Dom}(k) \text{ and } -1 \le k(x) \le 1.}$$

Example Find the domain and derivative of $\cos^{-1}(x^2-1)$

Domain: $-1 \le x^2 - 1 \le 1$ or $0 \le x^2 \le 2$ or $-\sqrt{2} \le x \le \sqrt{2}$.

Using the formula above with $k(x) = x^2 - 1$, we get

$$\frac{d}{dx}\cos^{-1}(x^2-1) = \frac{-2x}{\sqrt{1-(x^2-1)^2}}$$

Example

$$\int \frac{\cos^{-1}x}{\sqrt{1-x^2}}\,dx$$

Let $u = \cos^{-1}x$, $du = \frac{-1}{\sqrt{1-x^2}dx}$ or $dx = -\sqrt{1-x^2}du$. We get

$$\int \frac{\cos^{-1}x}{\sqrt{1-x^2}}\,dx = -\int u du = \frac{-u^2}{2} + C = \frac{-(\cos^{-1}x)^2}{2} + C$$