Lecture 7 : Indeterminate Forms

Recall that we calculated the following limit using geometry in Calculus 1:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Definition An indeterminate form of the type $\frac{0}{0}$ is a limit of a quotient where both numerator and denominator approach 0.

Example

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x} \qquad \lim_{x \to \infty} \frac{x^{-2}}{e^{-x}} \qquad \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$$

Definition An indeterminate form of the type $\frac{\infty}{\infty}$ is a limit of a quotient $\frac{f(x)}{g(x)}$ where $f(x) \to \infty$ or $-\infty$ and $g(x) \to \infty$ or $-\infty$.

Example

$$\lim_{x \to \infty} \frac{x^2 + 2x + 1}{e^x} \qquad \lim_{x \to 0^+} \frac{x^{-1}}{\ln x}.$$

L'Hospital's Rule Suppose *lim* stands for any one of

$$\lim_{x\to a} \qquad \lim_{x\to a^+} \qquad \lim_{x\to a^-} \qquad \lim_{x\to\infty} \qquad \lim_{x\to-\infty}$$

and $\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

If $lim\, \frac{f'(x)}{g'(x)}$ is a finite number L or is $\pm\infty$, then

$$\boxed{lim\, \frac{f(x)}{g(x)} \quad = \quad lim\, \frac{f'(x)}{g'(x)}.}$$

(Assuming that $f(x)$ and $g(x)$ are both differentiable in some open interval around a or ∞ (as appropriate) except possible at a, and that $g'(x) \neq 0$ in that interval).

Definition $lim\, f(x)g(x)$ is **an indeterminate form of the type** $0 \cdot \infty$ if

$$lim\, f(x) = 0 \quad \text{and} \quad lim\, g(x) = \pm\infty.$$

Example $\lim_{x\to\infty} x\tan(1/x)$

We can convert the above indeterminate form to an indeterminate form of type $\frac{0}{0}$ by writing

$$f(x)g(x) = \frac{f(x)}{1/g(x)}$$

or to an indeterminate form of the type $\frac{\infty}{\infty}$ by writing

$$f(x)g(x) = \frac{g(x)}{1/f(x)}.$$

We them apply L'Hospital's rule to the limit.

Indeterminate Forms of the type 0^0, ∞^0, 1^∞.

Type	Limit		
0^0	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = 0$	$lim\ g(x) = 0$
∞^0	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = \infty$	$lim\ g(x) = 0$
1^∞	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = 1$	$lim\ g(x) = \infty$

Example $\lim_{x \to 0}(1 + x^2)^{\frac{1}{x}}$.

Method

1. Look at $lim\ \ln[f(x)]^{g(x)} = lim\ g(x)\ln[f(x)]$.

2. Use L'Hospital to find $lim\ g(x)\ln[f(x)] = \alpha$. ($\alpha$ might be finite or $\pm\infty$ here.)

3. Then $lim f(x)^{g(x)} = lim\ e^{\ln[f(x)]^{g(x)}} = e^\alpha$ since e^x is a continuous function. (where e^∞ should be interpreted as ∞ and $e^{-\infty}$ should be interpreted as 0.)

Indeterminate Forms of the type $\infty - \infty$ occur when we encounter a limit of the form $lim(f(x) - g(x))$ where $lim\ f(x) = lim\ g(x) = \infty$ or $lim\ f(x) = lim\ g(x) = -\infty$

Example $\lim_{x \to 0^+} \frac{1}{x} - \frac{1}{\sin x}$

To deal with these limits, we try to convert to the previous indeterminate forms by adding fractions etc...

Cautionary tales L'Hospital's rule doesn't apply to every situation, and if used incorrectly, can sometimes give you the wrong answer.

Example

$$\lim_{x\to\infty} \frac{\sin x}{\ln x}$$

Example

$$\lim_{x\to\infty} \frac{x}{\left(\frac{1}{x}\right)}$$

Example

$$\lim_{x\to 0^+} \frac{x+1}{\sin x}$$

Try these Extra Fun Examples over Lunch

$$\lim_{x\to -\infty} \frac{2^x}{\sin(\frac{1}{x})}$$

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x}$$

$$\lim_{x\to\infty} x\tan(1/x)$$

$$\lim_{x\to 0^+} (e^{2x}-1)^{\frac{1}{\ln x}}$$

$$\lim_{x\to 1} (x)^{\frac{1}{x-1}}$$

Extra Examples: Indeterminate Forms

$$\lim_{x \to -\infty} \frac{2^x}{\sin(\frac{1}{x})}$$

(Note: You could use the sandwich theorem from Calc 1 for this if you prefer.)
This is an indeterminate form of type $\frac{0}{0}$. By L'Hospitals rule it equals:

$$\lim_{x \to -\infty} \frac{(\ln 2)2^x}{-\frac{1}{x^2}\cos(\frac{1}{x})} = \lim_{x \to -\infty} \frac{\ln 2}{\cos(\frac{1}{x})} \lim_{x \to -\infty} \frac{2^x}{-\frac{1}{x^2}}$$

$$= (\ln 2) \lim_{x \to -\infty} \frac{-x^2}{2^{-x}}$$

Applying L'Hospital again, we get that this equals

$$(\ln 2) \lim_{x \to -\infty} \frac{-2x}{-(\ln 2)2^{-x}}$$

Applying L'Hospital a third time, we get that this equals

$$\frac{2(\ln 2)}{\ln 2} \lim_{x \to -\infty} \frac{1}{-(\ln 2)2^{-x}} = 0$$

$$\lim_{x \to 0^+} \frac{\ln x}{\csc x}$$

This is an indeterminate form of type $\frac{\infty}{\infty}$
Applying L'Hospital's rule we get that it equals

$$\lim_{x \to 0^+} \frac{1/x}{\frac{-\cos x}{\sin^2 x}} = \lim_{x \to 0^+} \frac{-\sin^2 x}{x \cos x}$$

$$= \lim_{x \to 0^+} \frac{-\sin^2 x}{x} \lim_{x \to 0^+} \frac{1}{\cos x} = \lim_{x \to 0^+} \frac{-\sin^2 x}{x}$$

We can apply L'Hospital's rule again to get that the above limit equals

$$\lim_{x \to 0^+} \frac{-2 \sin x \cos x}{1} = 0$$

$$\lim_{x \to \infty} x \tan(1/x)$$

Rearranging this, we get an indeterminate form of type $\frac{0}{0}$

$$\lim_{x \to \infty} \frac{\tan(1/x)}{1/x} = \lim_{x \to \infty} \frac{\frac{-1}{x^2}\sec^2(1/x)}{\frac{-1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{1}{\cos^2(1/x)} = 1$$

$$\lim_{x\to 0^+}(e^{2x}-1)^{\frac{1}{\ln x}}$$

is an indeterminate form of type 0^0. Using continuity of the exponential function, we get $\lim_{x\to 0^+}(e^{2x}-1)^{\frac{1}{\ln x}} = e^{\lim_{x\to 0^+}\ln((e^{2x}-1)^{\frac{1}{\ln x}})} = e^{\lim_{x\to 0^+}\frac{1}{\ln x}\ln(e^{2x}-1)}$

For

$$\lim_{x\to 0^+}\frac{\ln(e^{2x}-1)}{\ln x}$$

we apply L'Hospital to get:

$$\lim_{x\to 0^+}\frac{\ln(e^{2x}-1)}{\ln x} = \lim_{x\to 0^+}\frac{\frac{(2e^{2x})}{e^{2x}-1}}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{(2xe^{2x})}{e^{2x}-1}$$

We apply L'Hospital again to get :

$$= \lim_{x\to 0^+}\frac{(2(e^{2x}+2xe^{2x}))}{2e^{2x}} = 1$$

Substiuting this into the original limit, we get

$$\lim_{x\to 0^+}(e^{2x}-1)^{\frac{1}{\ln x}} = e^1 = e$$

$$\lim_{x\to 1}(x)^{\frac{1}{x-1}}$$

$$\lim_{x\to 1}(x)^{\frac{1}{x-1}} = e^{\lim_{x\to 1}\ln(x^{\frac{1}{x-1}})} = e^{\lim_{x\to 1}\ln(x^{\frac{1}{x-1}})}$$

Focusing on the power we get

$$\lim_{x\to 1}\ln(x^{\frac{1}{x-1}}) = \lim_{x\to 1}\frac{\ln x}{x-1}$$

This is an indeterminate form of type $\frac{0}{0}$ so we can apply L'Hospital's rule to get

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{1/x}{1} = 1$$

Substitution this for the power of e above we get

$$\lim_{x\to 1}(x)^{\frac{1}{x-1}} = e^1 = e$$

$$\lim_{x\to\infty}\sqrt{x^2+\ln x}-x$$

This is an indeterminate form of type $\infty-\infty$

$$= \lim_{x\to\infty}(\sqrt{x^2+\ln x}-x)\frac{\sqrt{x^2+\ln x}+x}{\sqrt{x^2+\ln x}+x} = \lim_{x\to\infty}\frac{x^2+\ln x-x^2}{\sqrt{x^2+\ln x}+x}$$

$$= \lim_{x\to\infty}\frac{\ln x}{\sqrt{x^2+\ln x}+x}$$

This in an indeterminate form of type $\frac{\infty}{\infty}$. We can apply L'Hospital to get

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x^2+\ln x}+x} = \lim_{x\to\infty} \frac{1/x}{\frac{2x+1/x}{2\sqrt{x^2+\ln x}}+1}$$

We calculate

$$\lim_{x\to\infty} \frac{2x+1/x}{2\sqrt{x^2+\ln x}}$$

by dividing the numerator and denominator by x to get

$$\lim_{x\to\infty} \frac{2x+1/x}{2\sqrt{x^2+\ln x}} = \lim_{x\to\infty} \frac{2+1/x^2}{2\sqrt{1+\frac{\ln x}{x^2}}}$$

Applying L'Hospital to get $\lim \ln x/x^2 = \lim 1/2x^2 = 0$, we get

$$\lim_{x\to\infty} \frac{2x+1/x}{2\sqrt{x^2+\ln x}} = 1$$

and using this, we get

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x^2+\ln x}+x} = \lim_{x\to\infty} \frac{1/x}{\frac{2x+1/x}{2\sqrt{x^2+\ln x}}+1} = \frac{0}{2} = 0$$

Now this gives

$$\boxed{\lim_{x\to\infty} \sqrt{x^2+\ln x} - x = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x^2+\ln x}+x} = 0}$$