Lecture 9 : Trigonometric Integrals

Mixed powers of $\sin$ **and** $\cos$

Strategy for integrating

$$\boxed{\int \sin^m x \cos^n x dx}$$

We use substitution:

If **n is odd** use substitution with $u = \sin x$, $du = \cos x dx$ and convert the remaining factors of cosine using $\cos^2 x = 1 - \sin^2 x$. This will work even if $m = 0$.

Example

$$\int \sin^5 x \cos^3 x dx$$

If **m is odd** use substitution with $u = \cos x$, $du = -\sin x dx$ and convert the remaining factors of cosine using $\sin^2 x = 1 - \cos^2 x$. This will work if $n = 0$.

Example

$$\int \sin^3 x \cos^4 x dx$$

If **both powers are even** we reduce the powers using the half angle formulas:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Alternatively, you can switch to powers of sine and cosine using $\cos^2 x + \sin^2 x = 1$ and use the reduction formulas from the previous section.

Example

$$\int \sin^2 x \cos^2 x dx$$

Powers of $\tan$ **and** $\sec$. Strategy for integrating

$$\boxed{\int \sec^m x \tan^n x dx}$$

If m **is even** and $m > 0$, use substitution with $u = \tan x$, and use one factor of $\sec^2 x$ for $du = \sec^2 dx$. Use $\sec^2 x = 1 + \tan^2 x$ to convert the remaining factors of $\sec^2 x$ to a function of $u = \tan x$. This works even if $n = 0$ as long as $m \geq 4$.

Example $\int \sec^4 x \tan x dx$

If n is odd and $m \geq 1$ use substitution with $u = \sec x$, $du = \sec x \tan x \; dx$, and convert remaining powers of tan to a function of u using $\tan^2 x = \sec^2 x - 1$. This works as long as $m \geq 1$.

Example $\int \sec^3 x \tan x dx$.

If m odd and n is even we can reduce to powers of secant using the identity $\sec^2 x = 1 + \tan^2 x$.

Example $\int \sec x \tan^2 x dx$ (see integral of $\sec x$ and $\sec^3 x$ below.)

To evaluate

$$\boxed{\int \sin(mx)\cos(nx)dx \qquad \int \sin(mx)\sin(nx)dx \qquad \int \cos(mx)\cos(nx)dx}$$

we reverse the identities

$$\sin((m-n)x) = \sin(mx)\cos(nx) - \cos(mx)\sin(nx)$$

$$\sin((m+n)x) = \sin(mx)\cos(nx) + \cos(mx)\sin(nx)$$
$$\cos((m-n)x) = \cos(mx)\cos(nx) + \sin(nx)\sin(mx)$$
$$\cos((m+n)x) = \cos(mx)\cos(nx) - \sin(nx)\sin(mx)$$

to get

$$\sin(mx)\cos(nx) = \frac{1}{2}\big[\sin((m-n)x) + \sin((m+n)x\big]$$
$$\sin(mx)\sin(nx) = \frac{1}{2}\big[\cos((m-n)x) - \cos((m+n)x\big]$$
$$\cos(mx)\cos(nx) = \frac{1}{2}\big[\cos((m-n)x) + \cos((m+n)x\big]$$

Example $\int \sin 7x \cos 3x dx$

We have the following results for **powers of secant**

Example

$$\int \sec^0 x dx = \int 1 dx = x + C.$$

Example

$$\int \sec x dx = \ln|\sec x + \tan x| + C$$

Proof

$$\int \sec x dx = \int \sec x \Big(\frac{\sec x + \tan x}{\sec x + \tan x}\Big) dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

Using the substitution $u = \sec x + \tan x$, we get $du = \sec^2 x + \sec x \tan x$ giving us that the above integral is

$$\int \frac{1}{u} du = \ln|u| = \ln|\sec x + \tan x| + C.$$

Example

$$\int \sec^3 x dx = \int \sec^2 x \sec x \, dx$$

use integration by parts with $u = \sec x, \quad dv = \sec^2 x dx$ to get (a recurring integral)

$$\int \sec^3 x dx = \int \sec^2 x \sec x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx = \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

Solving for $\int \sec^3 x \, dx$, we get

$$\int \sec^3 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec^1 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln|\sec x + \tan x| + C.$$

In fact for $n \geq 3$, we can derive a reduction formula for powers of sec in this way:

$$\boxed{\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx.}$$

Powers of tangent can be reduced using the formula $\tan^2 x = \sec^2 x - 1$

Example

$$\int \tan^0 x dx = \int 1 dx = x + C.$$

Example

$$\int \tan x dx = \ln|\sec x| + C$$

Proof

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

Using the substitution $u = \cos x$, we get $du = -\sin x$ giving us that the above integral is

$$\int \frac{-1}{u} du = -\ln|u| = \ln|\sec x| + C.$$

Example

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$$

Example

$$\int \tan^3 x dx = \int (\sec^2 x - 1) \tan x dx = \int (\sec^2 x) \tan x dx - \int \tan x dx$$

$$= \frac{\tan^2 x}{2} + \ln|\sec x| + C.$$

In fact for $n \geq 2$, we can derive a reduction formula for powers of $\tan x$ using this method:

$$\boxed{\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx}$$

A few more examples

$$\int \cos^3 x \sin^3 x \, dx$$

$$\int \cos^2 x \tan^4 x \, dx$$

$$\int \sec^3 x \tan^3 x \, dx$$