$\int \sin^m x \cos^n x dx$, where n is odd.

Strategy for integrating

$$\boxed{\int \sin^m x \cos^n x dx}$$

We use substitution:
If **n is odd** (that is if the power of cosine is odd) we can use substitution with $u = \sin x$, $du = \cos x dx$ and convert the remaining factors of cosine using $\cos^2 x = 1 - \sin^2 x$. This will work even if $m = 0$.

Example

$$\int \sin^5 x \cos^3 x dx$$

- *Let* $u = \sin(x)$, $\quad du = \cos x \; dx$, $\quad \cos^2 x = 1 - \sin^2 x = 1 - u^2$.
- $\int \sin^5 x \cos^3 x dx \; = \int \sin^5 x \cos^2 x \cos x \; dx = \int (\sin^5 x)(1 - \sin^2 x) \cos x \; dx$.
- $= \int u^5(1 - u^2) \; du = \int u^5 - u^7 \; du = \frac{u^6}{6} - \frac{u^8}{8} + C = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C$.

$\int \sin^m x \cos^n x dx$, where n is odd.

The substitution $u = \sin x$ works even if $m = 0$ and we have an odd power of cosine.

Example $\int \cos^5 x\, dx$.

- $\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (\cos^2 x)^2 \cos x\, dx = \int (1 - \sin^2 x)^2 \cos x\, dx$.
- *Let* $u = \sin x, \quad du = \cos x\, dx$.
- *Then* $\int \cos^5 x\, dx = \int (1 - \sin^2 x)^2 \cos x\, dx = \int (1 - u^2)^2\, du = \int 1 - 2u^2 + u^4\, du$

$$= u - 2\frac{u^3}{3} + \frac{u^5}{5} + C = \sin x - 2\frac{\sin^3 x}{3} + \frac{\sin^5 x}{5} + C.$$

$\int \sin^m x \cos^n x dx$, where m is odd.

$$\int \sin^m x \cos^n x dx$$

If **m is odd** (that is if the power of sine is odd) we can use substitution with $u = \cos x$, $du = -\sin x dx$ and convert the remaining factors of sine using $\sin^2 x = 1 - \cos^2 x$. This will work even if $n = 0$.

Example $\int \sin^5 x \cos^4 x dx$. (Note that the power of cosine is even here, so the substitution $u = \sin x$ will not work.)

- *Let* $u = \cos(x)$, $\quad du = -\sin x\, dx$, $\quad \sin^2 x = 1 - \cos^2 x = 1 - u^2$.
- $\int \sin^5 x \cos^4 x dx = \int \sin^4 x \cos^4 x \sin x\, dx = \int (\sin^2 x)^2 (\cos^4 x) \sin x\, dx$.
- $= \int (1 - \cos^2 x)^2 \cos^4 x \sin x\, dx = \int (1 - u^2)^2 u^4 (-1\, du) = -\int (1 - 2u^2 + u^4) u^4\, du$
- $= -\int u^4 - 2u^6 + u^8\, du = -\left[\frac{u^5}{5} - 2\frac{u^7}{7} + \frac{u^9}{9}\right] + C$
- $= -\left[\frac{\cos^5 x}{5} - 2\frac{\cos^7 x}{7} + \frac{\cos^9 x}{9}\right] + C$.
- *Note the substitution* $u = \cos x$ *will work for odd powers of the sine function. See for example* $\int \sin^3 x\, dx$ *in the extra examples at the end of your notes.*

$\int \sin^m x \cos^n x dx$, where both m and n are even.

$$\int \sin^m x \cos^n x dx$$

If **both powers are even** we reduce the powers using the half angle formulas:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Alternatively, you can switch to powers of sine and cosine using $\cos^2 x + \sin^2 x = 1$ and use the reduction formulas from the previous section.

Example $\int \sin^2 x \cos^2 x dx$.

- $\int \sin^2 x \cos^2 x dx \ = \int [\frac{1}{2}(1 - \cos 2x)][\frac{1}{2}(1 + \cos 2x)] \, dx = \frac{1}{4} \int [1 - \cos^2(2x)] \, dx$
- *Now we can use the half angle formula again:* $\cos^2(2x) = \frac{1}{2}(1 + \cos 4x)$.
- $\int \sin^2 x \cos^2 x dx = \frac{1}{4} \int \left(1 - \cos^2(2x)\right) dx = \frac{1}{4} \int \left(1 - [\frac{1}{2}(1 + \cos 4x)]\right) dx$
- $= \frac{1}{4} \int \frac{1}{2} - \frac{1}{2} \cos 4x \, dx = \frac{1}{8} \int 1 - \cos 4x \, dx = \frac{x}{8} - \frac{\sin 4x}{32} + C$
- *See also the examples* $\int \sin^4 x \cos^2 x \, dx$ *and* $\int \sin^2 x \, dx$ *in the extra problems at the end of your notes.*

$\int \sin^m x \cos^n x dx$, where both m and n are even.

$$\int \sin^m x \cos^n x dx$$

Note If **both powers are even**, as an alternative to using the half angle formulas, you can switch to powers of sine and cosine using $\cos^2 x + \sin^2 x = 1$ and use the reduction formulas which can be derived using integration by parts:

$$\boxed{\int \cos^n x = \frac{1}{n}[\cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx]}$$

$$\boxed{\int \sin^n x dx = \frac{-1}{n}[\sin^{n-1} x \cos x - (n-1)\int \sin^{n-2} x dx]}$$

Example $\int \sin^2 x \cos^2 x dx$.

- $\int \sin^2 x \cos^2 x dx \ = \int [1 - \cos^2 x][\cos^2 x] \, dx = \int [\cos^2 x - \cos^4(x)] \, dx$
- *We can then integrate* $\cos^2 x$ *using the half angle formula and reduce the integral of* $\cos^4 x$ *to that of* $\cos^2 x$ *using the reduction formula above.*

$\int \sec^m x \tan^n x dx$

Strategy for integrating

$$\int \sec^m x \tan^n x dx$$

If m is even and $m > 0$, use substitution with $u = \tan x$, and use one factor of $\sec^2 x$ for $du = \sec^2 x\ dx$. Use $\sec^2 x = 1 + \tan^2 x$ to convert the remaining factors of $\sec^2 x$ to a function of $u = \tan x$. This works even if $n = 0$ as long as $m \geq 4$.

Example $\int \sec^4 x \tan x dx$

- $\int \sec^4 x \tan x dx \ = \int \sec^2 x \sec^2 x \tan x\ dx.$
- *Let* $u = \tan x, \quad du = \sec^2 x\ dx, \quad \sec^2 x = 1 + \tan^2 x.$
- $\int \sec^2 x \sec^2 x \tan x\ dx = \int [1 + \tan^2 x] \tan x \sec^2 x\ dx = \int [1 + u^2] u\ du$
- $= \int [u + u^3]\ du = \frac{u^2}{2} + \frac{u^4}{4} + C = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C.$

$\int \sec^m x \tan^n x dx$

Strategy for integrating

$$\boxed{\int \sec^m x \tan^n x dx}$$

If n **is odd and** $m \geq 1$ use substitution with $u = \sec x$, $du = \sec x \tan x \, dx$, and convert remaining powers of tan to a function of u using $\tan^2 x = \sec^2 x - 1$. This works as long as $m \geq 1$.

Example $\int \sec^3 x \tan x dx$.

- $\int \sec^3 x \tan x dx = \int \sec^2 x \sec x \tan x \, dx.$
- *Let* $u = \sec x, \quad du = \sec x \tan x \, dx.$
- $\int \sec^2 x \sec x \tan x \, dx = \int u^2 \, du = \frac{u^3}{3} + C$
- $= \frac{\sec^3 x}{3} + C.$
- *See also* $\int \sec^3 x \tan^5 x \, dx$ *in the extra examples.*

$\int \sec^m x \tan^n x dx$

Strategy for integrating

$$\boxed{\int \sec^m x \tan^n x dx}$$

If m **odd and** n **is even** we can reduce to powers of secant using the identity $\sec^2 x = 1 + \tan^2 x$.

Example $\int \sec x \tan^2 x dx$ (see integral of $\sec x$ and $\sec^3 x$ below.)

- $\int \sec x \tan^2 x dx = \int \sec x[\sec^2 x - 1]\, dx = \int \sec^3 x - \sec x\, dx.$
- *You will see how to calculate these integrals in the "powers of Secant" section below.*
- *See also* $\int \sec^3 x \tan^2 x\, dx$ *in the extra examples.*

$\int \sin(mx)\cos(nx)dx, \quad \int \sin(mx)\sin(nx)dx, \quad \int \cos(mx)\cos($

To evaluate

$$\int \sin(mx)\cos(nx)dx \qquad \int \sin(mx)\sin(nx)dx \qquad \int \cos(mx)\cos(nx)dx$$

we reverse the identities

$$\sin((m-n)x) = \sin(mx)\cos(nx) - \cos(mx)\sin(nx)$$
$$\sin((m+n)x) = \sin mx \cos nx + \sin nx \cos mx$$
$$\cos((m-n)x) = \cos(mx)\cos(nx) + \sin(nx)\sin(mx)$$
$$\cos((m+n)x) = \cos(mx)\cos(nx) - \sin(nx)\sin(mx)$$

to get

$$\sin(mx)\cos(nx) = \frac{1}{2}\left[\sin((m-n)x) + \sin((m+n)x\right]$$
$$\sin(mx)\sin(nx) = \frac{1}{2}\left[\cos((m-n)x) - \cos((m+n)x\right]$$
$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos((m-n)x) + \cos((m+n)x\right]$$

$\int \sin(mx)\cos(nx)dx$.

Example $\int \sin 7x \cos 3x dx$

- *We use* $\sin(mx)\cos(nx) = \frac{1}{2}\big[\sin((m-n)x) + \sin((m+n)x\big]$.
- $\int \sin 7x \cos 3x dx = \frac{1}{2}\int \sin(4x) + \sin(10x)\ dx$
- $= \frac{-1}{2}\left[\frac{\cos(4x)}{4} + \frac{\cos(10x)}{10}\right] + C.$
- *Also see* $\int \cos(8x)\cos(2x)\ dx$ *and* $\int \sin(x)\sin(2x)\ dx$ *in the extra examples.*

Powers of Secant

Example

$$\int \sec x dx = \ln|\sec x + \tan x| + C$$

- $\int \sec x dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$
- *Using the substitution* $u = \sec x + \tan x$, *we get* $du = (\sec^2 x + \sec x \tan x)\, dx$ *giving us that the above integral is*

$$\int \frac{1}{u} du = \ln|u| = \ln|\sec x + \tan x| + C.$$

$$\int \sec^2 x\, dx = \tan x + C.$$

Powers of Secant

Example

$$\int \sec^3 x dx$$

- *We use integration by parts with* $u = \sec x, \quad dv = \sec^2 x \, dx$. *We get* $du = \sec x \tan x \, dx$ *and* $v = \tan x$.
- $\int \sec^3 x dx = \int \sec^2 x \sec x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx$
- $= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$
- *Solving for* $\int \sec^3 x \, dx$, *we get*

$$\int \sec^3 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec^1 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln|\sec x + \tan x| + C.$$

- *In fact for* $n \geq 3$, *we can derive a reduction formula for powers of* $\sec$ *in this way:*

$$\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx.$$

Powers of Tangent

$$\int \tan^n x dx$$

- $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$
- *Using the substitution* $u = \cos x$, *we get* $du = -\sin x$ *we get*
- $\int \frac{\sin x}{\cos x} dx = \int \frac{-1}{u} du = -\ln|u| = \ln|\sec x| + C.$
- $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$
- $\int \tan^3 x dx = \int (\sec^2 x - 1)\tan x dx = \int (\sec^2 x)\tan x dx - \int \tan x dx$
  $= \frac{\tan^2 x}{2} - \ln|\sec x| + C = \frac{\tan^2 x}{2} + \ln|\cos x| + C.$
- *In fact for* $n \geq 2$, *we can derive a reduction formula for powers of* $\tan x$ *using this method:*

$$\boxed{\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx}$$