(1) Let X be the locus

$$X = \{(z, w) \in \mathbb{C}^2 \mid z = e^w\}.$$

Show that X is a Riemann surface, either by exhibiting a collection of holomorphic charts or else by appealing to other technology.

(2) Recall that a smooth manifold M is said to be *orientable* if there is an atlas $\{U_i, \phi_i\}$ such that each transition function ϕ_{ij} has everywhere positive Jacobian determinant. Prove that every Riemann surface is orientable.

(3) (*Projectivization*) Here is an alternative way to define a compactification of the solution set to $y^2 = p(x)$ in $\mathbb{C}^2$, where $p(x)$ is a polynomial with distinct roots of degree three or four.

(a) Let $F(x, y, z)$ be a homogeneous polynomial[1] of degree $d \geq 1$. Verify that the set

$$V(F) = \{[x : y : z] \in \mathbb{CP}^2 \mid F(x, y, z) = 0\}$$

is well-defined. Why is the hypothesis of homogeneity necessary?

(b) Let $F(x, y, z)$ be homogeneous. Show that if at each point $[x : y : z] \in V(F)$, at least one of the partial derivatives F_x, F_y, F_z is nonvanishing, then $V(F)$ is a Riemann surface.

(c) Show that any such $V(F)$ is compact.

(d) Let $f(x, y)$ be a polynomial of degree d, not necessarily homogeneous. Define the *homogenization* of f to be the homogeneous polynomial $F(x, y, z)$ obtained from f by adding in multiples of z to each term so as to make the result homogeneous of degree d. Verify that for $f(x, y) = y^2 - p(x)$ as in the problem statement, the resulting homogenization F satisfies the condition of (b) and hence $V(F)$ is a compact Riemann surface.

(e) Describe an embedding of $V(f) \subset \mathbb{C}^2$ into $V(F) \subset \mathbb{CP}^2$. What is $V(F) \setminus V(f)$ (with respect to this embedding)?

(f) What goes wrong if p is of degree greater than four?

(4) Show that for each $d \geq 1$, there is a complex torus $X = \mathbb{C}/\Lambda$ and an analytic map $f : X \to X$ of degree d. [If you get stuck, contemplate the relationship between pieces of paper of size A4 and A5.]

(5) Give an explicit example of a polynomial $p(x)$ of degree 6 such that the compact Riemann surface defined by the equation $y^2 = p(x)$ admits a nonconstant holomorphic map to a Riemann surface of genus one. Describe the branch locus, and verify that the Riemann-Hurwitz formula is satisfied.

[1] Recall a polynomial is *homogeneous* of degree d if every monomial has total degree d (counting each variable as having degree 1)