(1) Let X be the locus

$$X = \{(z, w) \in \mathbb{C}^2 \mid z = e^w\}.$$

Show that X is a Riemann surface, either by exhibiting a collection of holomorphic charts or else by appealing to other technology.

Solution: The fastest way to do this is to appeal to the regular value theorem (RVT). Let $F : \mathbb{C}^2 \to \mathbb{C}$ be the holomorphic function given by $F(z, w) = z - e^w$. Then X is given as

$$X = F^{-1}(0),$$

and if 0 is a regular value of F, then X will be a holomorphic submanifold of $\mathbb{C}^2$, i.e. a Riemann surface.

To check that this holds, it is necessary to verify that at every point of X, at least one of the partial derivative F_z or F_w is nonvanishing. As $F_z = 1$ is *everywhere nonvanishing*, the claim follows.

[Note: Also $F_w = e^w$ is everywhere nonvanishing, but this is something of a red herring. One can replace e^w with *any* holomorphic function $g(w)$ and the same argument shows that the solution set $z = g(w)$ still defines a Riemann surface. The projection onto the w-factor is a globally well-defined holomorphic map that gives a global inverse to $g(w)$.]

(2) Recall that a smooth manifold M is said to be *orientable* if there is an atlas $\{U_i, \phi_i\}$ such that each transition function ϕ_{ij} has everywhere positive Jacobian determinant. Prove that every Riemann surface is orientable.

Solution: Let X be a Riemann surface. We must verify that for the atlas of charts $\{U_i, \phi_i\}$ realizing the Riemann surface structure on X, the transition functions $\phi_{ij} : \phi_i^{-1}(U_i \cap U_j) \to \mathbb{C}$ have positive Jacobian, upon identifying $\mathbb{C}$ with $\mathbb{R}^2$. This is a consequence of the Cauchy-Riemann equations. Let $f(z)$ be a holomorphic function, and write $f(x + iy) = u(x + iy) + iv(x + iy)$. Then the Cauchy-Riemann equations assert that

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

for all $x + iy$ in the domain of f. Thus, as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, the Jacobian of f is given as

$$\mathrm{Jac}(f)(x + iy) = (u_x v_y - u_y v_x)(x + iy) = (u_x^2 + u_y^2)(x + iy) \geq 0.$$

Moreover, the Jacobian is *strictly* positive whenever the complex derivative $f'(x + iy)$ is nonvanishing. Since $f = \phi_{ij}$ is a transition function, it is everywhere invertible, so that positivity of the Jacobian indeed holds.

(3) (*Projectivization*) Here is one way to define a compactification of the solution set to $y^2 = p(x)$ in $\mathbb{C}^2$, where $p(x)$ is a polynomial with distinct roots of degree three.

(a) Let $F(x, y, z)$ be a homogeneous polynomial[1] of degree $d \geq 1$. Verify that the set

$$V(F) = \{[x : y : z] \in \mathbb{CP}^2 \mid F(x, y, z) = 0\}$$

is well-defined. Why is the hypothesis of homogeneity necessary?

[1]Recall a polynomial is *homogeneous* of degree d if every monomial has total degree d (counting each variable as having degree 1)

[Important comment: other level sets "$F(x, y, z) = \lambda$" for $\lambda \neq 0$ *are not well-defined!* That is, *F is not a function on $\mathbb{CP}^2$!*]
Solution: We must check that the condition $F(x, y, z) = 0$ is independent of the choice of equivalence class representative, i.e. we must check that $F(x, y, z) = 0$ if and only if $F(\lambda x, \lambda y, \lambda z) = 0$ for all $\lambda \in \mathbb{C}^*$. Since F is homogeneous of degree d, it follows that $F(\lambda x, \lambda y, \lambda z) = \lambda^d F(x, y, z)$, so that this is indeed the case.

(b) Let $F(x, y, z)$ be homogeneous. Show that if at each point $[x : y : z] \in V(F)$, at least one of the partial derivatives F_x, F_y, F_z is nonvanishing, then $V(F)$ is a Riemann surface.
[Hint: you may find Euler's identity $F(x, y, z) = \frac{1}{d}(xF_x + yF_y + zF_z)$ useful; here F is a homogeneous polynomial of degree d.]
Solution: First, here's how *not* to do this problem. It's tempting to try and use the RVT: compute the partials F_x, F_y, F_z, and then check that the only point where all of F, F_x, F_y, F_z vanish is the "point" $[0 : 0 : 0]$, which isn't a point in projective space at all, hence the solution set $V(F)$ is everywhere smooth. This is not a valid argument because of the comment I made above: F does not define a holomorphic function on $\mathbb{CP}^2$ at all! In fact, by the maximum principle, since $\mathbb{CP}^2$ is compact, *every holomorphic function on $\mathbb{CP}^2$ is constant*.
The good news is that this idea can be repaired and made to work. Consider the coordinate patch $U_x \subset \mathbb{CP}^2$ defined as the set where $x \neq 0$, i.e. the set $U_x = \{[1 : y : z] \mid (y, z) \in \mathbb{C}^2\}$; define U_y and U_z likewise. Then $V(F)$ is given as the union of the pieces $V(F) \cap U_i$ for $i = x, y, z$. Identifying U_i with $\mathbb{C}^2$, we now *do* have that F defines a holomorphic function on U_i, and $V(F) \cap U_i$ is given as the zero level set of this function. I claim that each $V(F) \cap U_i$ is a Riemann surface. For instance, let's consider the coordinate patch U_z where $z \neq 0$ (the other coordinate patches can be treated identically). Every point in U_z can be written *uniquely* as $[x : y : 1]$ and so be identified with $(x, y) \in \mathbb{C}^2$. Here, F specializes to an actual (holomorphic) function $F^z : U_z \to \mathbb{C}$ by sending $[x : y : 1]$ to $F(x, y, 1)$. Now RVT says that $V(F) \cap U_z$ is a Riemann surface if there is no point on $V(F) \cap U_z$ where both partials of F^z vanish. Note $F^z_x(x, y) = F_x(x, y, 1)$ and similarly for F^z_y.
Potentially we could be in trouble: couldn't F and F_x and F_y all vanish at some such point but $F_z \neq 0$ there? A basic theorem on homogeneous polynomials shows that this can't happen:

Fact (Euler). If $F(x, y, z)$ is a homogeneous polynomial of degree d, then

$$F = \frac{1}{d}(xF_x + yF_y + zF_z).$$

This shows that if F, F_x, F_y all vanish at a point, so too does F_z. Thus, at least one of F^z_x, F^z_y must be nonvanishing for every point of $V(F) \cap U_z$. We have shown that $V(F) \cap U_z$ is a Riemann surface; the same result holds for the intersection of $V(F)$ with the other coordinate patches. The Riemann surface structures induced on points in the overlap are compatible, since the transition functions between coordinate patches are holomorphic.

(c) Show that any such $V(F)$ is compact.
Solution: Again, there's a fake proof of this which is very tempting but must be avoided. The fake proof goes like this: since $\mathbb{CP}^2$ is compact, it suffices to show that $V(F)$ is a closed subset. *If* $V(F)$ were realizable as a global level set, it would be the inverse image of a closed set (the singleton $\{0\} \subset \mathbb{C}$) under a continuous map F, and hence be closed, QED.
But of course $V(F)$ is not globally a level set. Luckily there is an easy fix: we can cover $V(F)$ by the three pieces $V(F) \cap U_i$ as above. Each of these *is* a level set for a function on U_i, and hence each $V(F) \cap U_i$ is a closed subset of U_i, and its complement is an open subset of U_i. Since each U_i is itself the complement of a closed set (the line determined by the vanishing of one of the coordinates, a copy of $\mathbb{CP}^1$), it is open in $\mathbb{CP}^2$ and hence the complement of $V(F) \cap U_i$ is open not just in U_i but moreover in $\mathbb{CP}^2$. The union of these three local complements is the complement to $V(F)$ in $\mathbb{CP}^2$, realizing it as the union of open sets, as desired.

(d) Let $f(x, y)$ be a polynomial of degree d, not necessarily homogeneous. Define the *homogenization* of f to be the homogeneous polynomial $F(x, y, z)$ obtained from f by adding in multiples of z to each term so as to make the result homogeneous of degree d. Verify that for $f(x, y) = y^2 - p(x)$ as in the problem statement, the resulting homogenization F satisfies the condition of (b) and hence $V(F)$ is a compact Riemann surface.
Solution: Write $p(x) = ax^3+bx^2+cx+d$, where $a \neq 0$. Then the homogenization is given by

$$F(x, y, z) = y^2z - ax^3 - bx^2z - cxz^2 - dz^3.$$

We compute

$$F_x = -3ax^2 - 2bxz - cz^2, \qquad F_y = 2yz, \qquad F_z = y^2 - bx^2 - 2cxz - 3dz^2.$$

Note that F_x is nothing more than the homogenization of the degree-two polynomial $f'(x)$. Thus, if $z \neq 0$, then we know from the affine situation (as done in class) that since p has simple roots, at least one of F_x and F_y is nonvanishing at every point of $V(F)$ with $z \neq 0$. It remains only to analyze the points with $z = 0$. Looking to F_x, we see that if $F_x = 0$ as well, also $x = 0$, and then (e.g. looking at F_z), also $y = 0$, so that no point in projective space satisfies all these constraints. We conclude that $V(F)$ satisfies the condition of (b) and hence is a compact Riemann surface.

(e) Describe an embedding of $V(f) \subset \mathbb{C}^2$ into $V(F) \subset \mathbb{CP}^2$. What is $V(F) \setminus V(f)$ (with respect to this embedding)?
Solution: Simply send (x, y) in $V(f)$ to $[x : y : 1]$. This lies on $V(F)$, since $F(x, y, 1) = f(x, y)$ by construction. The complement is the set of solutions to F with $z = 0$. From above, $F(x, y, 0) = -ax^3$, so there is a unique point $[0 : 1 : 0]$

of this form.

(f) What goes wrong if p is of degree greater than three?
Solution: If $\deg(p) > 3$, then the extra point at infinity will fail to be smooth (i.e. all partial derivatives will vanish). As an example, consider $p(x) = x^4 - 1$ (note that this has four distinct roots, so the solutions to $y^2 = x^4 - 1$ in $\mathbb{C}^2$ form a Riemann surface). The homogenized equation is

$$F(x, y, z) = y^2 z^2 - x^4 + z^4,$$

with

$$F_x = -4x^3, \qquad F_y = 2yz^2, \qquad F_z = 2y^2 z + 4z^3.$$

Note that $[0 : 1 : 0]$ still lies on $V(F)$, but now since y^2 was homogenized by adding in z^2 (since the highest degree in x is now 4), the term $2y^2 z$ in F_z automatically vanishes when $z = 0$, and in fact all of F_x, F_y, F_z vanish at this point.

(4) Show that for each $d \geq 1$, there is a complex torus $X = \mathbb{C}/\Lambda$ and an analytic map $f : X \to X$ of degree d. [If you get stuck, contemplate the relationship between pieces of paper of size A4 and A5.]
Solution: To do this problem correctly, you need to clearly define (a) which torus $X = \mathbb{C}/\Lambda$ you are using and (b) what the self-map is. For (a), define $\Lambda \subset \mathbb{C}$ as the lattice generated by 1 and $i\sqrt{d}$. The key observation is that a rectangle of aspect ratio $1 : \sqrt{d}$ decomposes into d subrectangles of the same aspect ratio (A4/A5 paper exhibits this in the case $d = 2$). These subrectangles are sideways and reduced in scale by a factor of $\sqrt{d}$. Thus the map $f : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto i\sqrt{d}z$ preserves the lattice Λ and hence descends to a map $\overline{f} : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$. The induced self-map of Λ has image of index d as a subgroup; this coincides with the degree as a (covering) map, as was to be shown.

(5) Give an explicit example of a polynomial $p(x)$ of degree 6 such that the compact Riemann surface defined by the equation $y^2 = p(x)$ admits a nonconstant holomorphic map to a Riemann surface of genus one. Describe the branch locus, and verify that the Riemann-Hurwitz formula is satisfied.
Solution: Let $p(x)$ be a cubic polynomial with distinct *nonzero* roots. Then $p(x^2)$ is a polynomial of degree 6, again with distinct roots (at the square roots of the roots of p). You can see that the map $\pi : (x, y) \mapsto (x^2, y)$ sends a point on $X = V(y^2 = p(x^2))$ to a point on $Y = V(y^2 = p(x))$, producing the desired map.

Let's think about Riemann-Hurwitz. First, the degree of π is 2: a general point $(x, y) \in Y$ has the two preimages $(\pm\sqrt{x}, y)$ on X. Branching happens where we have fewer preimages, i.e. where $x = 0$. There are two such points on X, the (two distinct) points $(0, \pm\sqrt{p(0)})$. Now Y has genus 1. To compute $\chi(X)$, we find

$$\chi(X) = 2(0) - 2(2 - 1) = -2,$$

showing that $g(X) = 2$.

Admittedly, this is a little fast and loose with what happens at infinity. Let's discuss what happens there. Remember that the curve defined by $y^2 = f(x)$ has a single point over infinity if $\deg f$ is odd (in which case it is a ramification point with $d_x = 2$), and if $\deg f$ is even, then there are two points over infinity, both unramified. Thus $y^2 = p(x^2)$ has two points over infinity, call them ∞_1 and ∞_2, while the target $y^2 = p(x)$ has just a single point, call it ∞. The map $(x, y) \mapsto (x^2, y)$ extends over the points at infinity by sending both ∞_1, ∞_2 to ∞. Thus we see that the preimage of ∞ consists of two points, so that neither point is ramified. This is consistent with the Riemann-Hurwitz analysis we did above: if there were any extra "hidden" branching at infinity, it would contribute in the above formula, pushing $\chi(X)$ down further. But $\chi(X) = -2$ is already achieved just from the two ramification points we already identified, so that no further ramification is possible.