Note: Throughout, X denotes a compact Riemann surface.

(1) Let X, Y be compact Riemann surfaces of genera $g(X), g(Y)$, respectively, and let $f : X \to Y$ be holomorphic. Prove the following:
   (a) If $g(Y) > g(X)$, then f is necessarily constant.
   (b) If $g(X) = g(Y) \geq 2$, then f is either constant or an isomorphism.

(2) (Uniqueness of branched covers) Let X, Y be compact Riemann surfaces, and let $f : X \to \widehat{\mathbb{C}}$ and $g : Y \to \widehat{\mathbb{C}}$ be nonconstant meromorphic functions. Suppose that (1) the branch sets $B(f) = B(g) = B \subset \widehat{\mathbb{C}}$ are equal and (2) the covering spaces $f : X^\circ \to \widehat{\mathbb{C}} - B$ and $g : Y^\circ \to \mathbb{C} - B$ are isomorphic as covering spaces. Show that $X \cong Y$ as Riemann surfaces. [Hint: if $X^\circ \cong Y^\circ$ as covering spaces, then there is a map $\iota : X^\circ \to Y^\circ$ that covers the identity on $\widehat{\mathbb{C}}$. Show that ι is holomorphic, then use the removable singularity theorem to extend to an isomorphism $X \to Y$.]

(3) (a) For which values of g does there exist a compact Riemann surface of genus g admitting a degree 5 map $f : X \to \mathbb{CP}^1$ with $B(f) \subset \{0, 1, \infty\}$ (this containment may be proper)? [Note: there may be more than one topological type of branched cover f for a given genus. I'm not asking you to enumerate them all, just to tell me about the possible genera.]
   (b) For each g which exists, choose one of the possible topological types and give an explicit pair of elements $\sigma_0, \sigma_1 \in S_5$ describing the monodromy of X over 0 and 1. (In other words, describe the associated map $\rho : \pi_1(\mathbb{CP}^1 - B(f)) \to S_5$ by giving its values on a loop around 0 and a loop around 1).
   (c) Show that there are exactly two covers that are Galois (i.e. regular in the sense of covering space theory, or equivalently that the field extension $\mathcal{M}(X)/\mathbb{C}(x)$ is Galois). The notion of isomorphism you should use is the following: covers $f : X \to \mathbb{CP}^1$ and $g : Y \to \mathbb{CP}^1$ are isomorphic if there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{CP}^1 & \xrightarrow[k]{} & \mathbb{CP}^1 \end{array}$$

   for which h and k are isomorphisms. In explicit terms, identify $\pi_1(\mathbb{CP}^1 \setminus B(f))$ with the free group with presentation $\langle x, y, z \mid xyz = 1\rangle$ (with x, y, z corresponding to the monodromies around $0, 1, \infty$). Then two monodromy homomorphisms are equivalent if they are equal after some permutation of x, y, z (corresponding to an automorphism of the base) and some conjugation automorphism of S_5 (corresponding to a covering space automorphism covering the identity).
   (d) Give explicit equations for each of the covers you found in (c).

(4) Find a Belyi polynomial $p(z)$ of degree 5 such that $p^{-1}([0,1])$ is homeomorphic to the letter Y, with the fork at $z = 0$. (The Belyi condition means that $p(0) = 0, p(1) = 1$, and the critical values of p are contained in $\{0, 1\}$.)