(1) (Filling in punctures) Here you will check some of the details of how "filling in punctures" actually works. The basic principle to exploit is the following:
Fact: Let $U \subset X$ be an open set in a Riemann surface that admits a coordinate $\phi : U \to \Delta^*$ identifying U with the punctured unit disk. Then U (and hence X) can be enlarged to U^+ by adjoining the missing point $\phi^{-1}(0)$. Since this new point lies in only one chart, there is no compatibility that needs to be checked.
  (a) The octahedron **O** admits a system of charts away from the vertices by unfolding pairs of adjacent faces. Extend the Riemann surface structure to all of **O**. [Hint: four equilateral triangles meet at the vertex, for a total angle of $4\pi/3$. Close this up to get an identification with Δ^* by applying the map $z^{3/2}$.]
  (b) (Uniqueness of branched covers, I) Let X be a compact topological surface and let Y be a compact Riemann surface. Suppose $f : X \to Y$ is a topological branched cover (i.e. there are finite sets $B \subset X$ and $D \subset Y$ such that f restricts to a genuine cover $X - B \to Y - D$). Define a Riemann surface structure on X for which f becomes holomorphic.

(2) (Uniqueness of branched covers, II) Let X, Y be compact Riemann surfaces, and let $f : X \to \widehat{\mathbb{C}}$ and $g : Y \to \widehat{\mathbb{C}}$ be nonconstant meromorphic functions. Suppose that (1) the branch sets $B(f) = B(g) = B \subset \widehat{\mathbb{C}}$ are equal and (2) the covering spaces $f : X^\circ \to \widehat{\mathbb{C}} - B$ and $g : Y^\circ \to \mathbb{C} - B$ are isomorphic as covering spaces. Show that $X \cong Y$ as Riemann surfaces. [Hint: if $X^\circ \cong Y^\circ$ as covering spaces, then there is a map $\iota : X^\circ \to Y^\circ$ that covers the identity on $\widehat{\mathbb{C}}$. Show that ι is holomorphic, then use the removable singularity theorem to extend to an isomorphism $X \to Y$.]

(3) Find a Belyi polynomial $p(z)$ of degree 5 such that $p^{-1}([0,1])$ is homeomorphic to the letter Y, with the fork at $z = 0$. (The Belyi condition means that $p(0) = 0, p(1) = 1$, and the critical values of p are contained in $\{0, 1\}$.)

(4) Prove that the coefficients of any Belyi polynomial are algebraic numbers.