(1) Let X, Y be compact Riemann surfaces of genera $g(X), g(Y)$, respectively, and let $f : X \to Y$ be holomorphic. Prove the following:
   (a) If $g(Y) > g(X)$, then f is necessarily constant.
   (b) If $g(X) = g(Y) \geq 2$, then f is either constant or an isomorphism.

Solution: Both of these are applications of Riemann-Hurwitz; let's start by examining the formula itself. If $f : X \to Y$ is nonconstant and holomorphic of degree d, then

$$2 - 2g(X) = d(2 - 2g(Y)) - \sum(d_x - 1).$$

Since the second term is nonpositive, we drop it to obtain an inequality, which can be rearranged to become

$$dg(Y) \leqslant (d-1) + g(X).$$

For (a), if $g(Y) > g(X)$, then the above shows that $(d-1)g(Y) < d - 1$, so that $g(Y) = 0$, but then $g(Y) > g(X) \geq 0$ cannot also hold. For (b), if $g(X) = g(Y) = g$, then the inequality becomes $(d-1)g \leqslant d - 1$. Under the assumption $g \geq 2$, we conclude that the only way this can hold is if f is constant (in which case RH doesn't apply) or else $d = 1$. Since the ramification indices d_x are constrained by the condition $1 \leqslant d_x \leqslant d$, we conclude that the ramification set is empty, so that f is a covering map of degree 1, hence an isomorphism.

(2) (Uniqueness of branched covers, II) Let X, Y be compact Riemann surfaces, and let $f : X \to \widehat{\mathbb{C}}$ and $g : Y \to \widehat{\mathbb{C}}$ be nonconstant meromorphic functions. Suppose that (1) the branch sets $B(f) = B(g) = B \subset \widehat{\mathbb{C}}$ are equal and (2) the covering spaces $f : X^\circ \to \widehat{\mathbb{C}} - B$ and $g : Y^\circ \to \mathbb{C} - B$ are isomorphic as covering spaces. Show that $X \cong Y$ as Riemann surfaces. [Hint: if $X^\circ \cong Y^\circ$ as covering spaces, then there is a map $\iota : X^\circ \to Y^\circ$ that covers the identity on $\widehat{\mathbb{C}}$. Show that ι is holomorphic, then think about how to extend this to a holomorphic map $X \to Y$.]

Solution: Following the hint, the hypotheses imply that the covering maps $f : X^\circ \to \widehat{\mathbb{C}} - B$ and $g : Y^\circ \to \widehat{\mathbb{C}} - B$ are isomorphic as covering spaces, so that there is a map $\iota : X^\circ \to Y^\circ$ covering the identity. We claim that ι is necessarily holomorphic. Since f, g are covering maps, the restriction of either one to a suitably small neighborhood is a coordinate on X° or Y° as appropriate. Let $U \subset X^\circ$ be some open set, sufficiently small so that f restricts to a homeomorphism onto its image, and sufficiently small so that g is a local homeomorphism over $f(U)$. Then the composition $g^{-1} \circ f$ gives a homeomorphism between U and $\iota(U)$. Using f and g as coordinates on X and Y respectively, this is nothing more than the identity, so is manifestly holomorphic.

We next claim that ι admits an extension $\iota : X \to Y$. To do so, we must specify where the missing points $f^{-1}(B) \subset X$ are sent. Let $b \in B$ be given, let $U \subset \widehat{\mathbb{C}}$ be a small neighborhood of b, and let $U^\circ := U \setminus \{b\}$. Then $f^{-1}(U)$ is a collection of $k \geq 1$ disks, and $f^{-1}(U^\circ)$ is a collection of k punctured disks, with the center point of each component deleted. Since ι realizes an isomorphism between the covering spaces X° and Y° of $\widehat{\mathbb{C}}$, it induces a bijection between the components of $f^{-1}(U)$ and $g^{-1}(U)$. Then ι extends over all of $f^{-1}(b)$ by sending the center point of each component of $f^{-1}(U)$ to the center of the corresponding component of $g^{-1}(U)$.

It remains to show that ι is holomorphic at points in the branch set. Let $x \in X$ be such a point, and let $d_x \geq 1$ be the local order of branching. Then x is at the center of a disk which locally maps d_x-to-one onto $\widehat{\mathbb{C}}$ under f, and by the previous paragraph, points near $\iota(x)$ map d_x-to-one onto $\widehat{\mathbb{C}}$ under g. By a result in class, there exist coordinates z near $x \in X$ and ξ near $f(x) = g(x)$ in $\widehat{\mathbb{C}}$ such that f is locally represented as $\xi = z^{d_x}$. Likewise, there are coordinates w near $\iota \in Y$ and η near $f(x) = g(x)$ in $\widehat{\mathbb{C}}$ such that g is represented as $\eta = w^{d_x}$. In these coordinates, ι is represented as the holomorphic transition function $\eta\xi^{-1}$, as desired.

This shows that $\iota : X \to Y$ is holomorphic. Since X°, Y° are isomorphic as covering spaces, there is an inverse $\iota^{-1} : Y^\circ \to X^\circ$; the same construction applies to show that this admits a holomorphic extension $\iota^{-1} : Y \to X$, and we conclude that $X \cong Y$ as Riemann surfaces.

(3) (a) For which values of g does there exist a compact Riemann surface of genus g admitting a degree 5 map $f : X \to \mathbb{CP}^1$ with $B(f) \subset \{0, 1, \infty\}$ (this containment may be proper)? [Note: there may be more than one topological type of branched cover f for a given genus. I'm not asking you to enumerate them all, just to tell me about the possible genera.]

(b) For each g which exists, choose one of the possible topological types and give an explicit pair of elements $\sigma_0, \sigma_1 \in S_5$ describing the monodromy of X over 0 and 1. (In other words, describe the associated map $\rho : \pi_1(\mathbb{CP}^1 - B(f)) \to S_5$ by giving its values on a loop around 0 and a loop around 1).

(c) Show that there are exactly two covers that are Galois (i.e. regular in the sense of covering space theory, or equivalently that the field extension $\mathcal{M}(X)/\mathbb{C}(x)$ is Galois). The notion of isomorphism you should use is the following: covers $f : X \to \mathbb{CP}^1$ and $g : Y \to \mathbb{CP}^1$ are isomorphic if there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ \mathbb{CP}^1 \arrow[r, "k"'] & \mathbb{CP}^1 \end{tikzcd}$$

for which h and k are isomorphisms. In explicit terms, identify $\pi_1(\mathbb{CP}^1 \setminus B(f))$ with the free group with presentation $\langle x, y, z \mid xyz = 1 \rangle$ (with x, y, z corresponding to the monodromies around $0, 1, \infty$). Then two monodromy homomorphisms are equivalent if they are equal after some permutation of x, y, z (corresponding to an automorphism of the base) and some conjugation automorphism of S_5 (corresponding to a covering space automorphism covering the identity).

(d) Give explicit equations for each of the covers you found in (c).

Solution: (a) Let's see what Riemann-Hurwitz tells us about this. We have

$$2-2g=5(2)-b,$$

where $b=\sum_{x\in X}(d_x-1)$. By hypothesis, f is branched only over $0,1,\infty$. Since f has degree 5, we see that the local branching behavior over each of these points corresponds to a partition of 5, recording the local orders of branching. There are thus the following possibilities for the orders of branching over each point:

$$5, 4+1, 3+2, 3+1+1, 2+1+1+1, 1+1+1+1+1.$$

Let us compute the contribution to b coming from each type of point. In the same order, these are

$$4, 3, 2+1=3, 2, 1, 0.$$

Therefore, b is the sum of three numbers between 0 and 4, although there are additional constraints. To see what these are, we rearrange the Riemann-Hurwitz equation to find

$$g=b/2-8.$$

Thus, b must be an even integer, and $b\geq 8$. There are thus three possibilities: $b=8,10,12$, corresponding respectively to $g=0,1,2$.

For (b), let's start with $b=12$. Here we must have a contribution of 4 over each of $0,1,\infty$; this means that the monodromy over each point must be a 5-cycle. Thus we must have σ_0,σ_1 be 5-cycles such that the product $\sigma_0\sigma_1$ giving the monodromy over ∞ is also a 5-cycle. One choice is to let $\sigma_1=\sigma_0$ an arbitrary 5-cycle, since the square of a 5-cycle is again a 5-cycle.

For $b=10$, we have options: the local monodromies can either contribute $4,4,2$ or else $4,3,3$. Let's work an example in the second case. Here, there is an additional possibility: the contribution 3 can arise either from a branch point of order 4, or from two branch points of order 3 and 2. In the first case, the local monodromy is a 4-cycle, while in the second it is a product of a 3-cycle and a 2-cycle. For concreteness, let's choose $\sigma_0=(12345)$ the 5-cycle. If we choose $\sigma_1=(1234)$, then $\sigma_0\sigma_1=(135)(24)$ is of this second type, so the ramification over ∞ is of the latter type.

Finally, for genus 0 (i.e. $b=8$), let's demonstrate an example with branching of type $3,3,2$. As before, 3 could come from one of two places; the contribution 2 must come from a single 3-cycle. There's a new subtlety here: we need to make sure that the subgroup of S_5 we build is *transitive* (i.e. acts transitively on the set $\{1,2,3,4,5\}$ of fibers), otherwise the covering space will be *disconnected*. Let's take $\sigma_0=(1234)$, and try and build a 4-cycle σ_1 such that $\sigma_0\sigma_1$ is a 3-cycle. Transitivity will be ensured as long as 5 appears in σ_1, since σ_0 already ensures transitivity among $\{1,2,3,4\}$. A little bit of trial and error gives that $\sigma_1=(5432)$ produces $\sigma_0\sigma_1=(125)$ as desired.

We consider (c). A covering will be Galois if it is classified as the kernel of a map $\pi_1(\widehat{\mathbb{C}}-B)\to G$. Equivalently, this is the case if the monodromy action is *free* (all points have trivial stabilizer subgroups), on top of being transitive. Since the degree must be 5, this forces $G=\mathbb{Z}/5\mathbb{Z}$. Thus we see that in terms of local monodromy, σ_0 and σ_1 must both live in some subgroup of S_5 isomorphic to $\mathbb{Z}/5$. Up to conjugacy there is a unique such subgroup, generated by some 5-cycle. Note that every element of this group is either a

5-cycle, contributing 4 to b, or else is trivial and contributes 0. In particular, only the values $b = 12$ and $b = 8$ are realizable.

To analyze all of the possibilities up to isomorphism, let's first identify $\mathbb{Z}/5\mathbb{Z} \leqslant S_5$ with the usual integers mod 5. We seek to classify the homomorphisms $\mu : F_2 \to \mathbb{Z}/5\mathbb{Z}$ up to the automorphism group of $\mathbb{Z}/5\mathbb{Z}$ (which is the multiplicative group $(\mathbb{Z}/5\mathbb{Z})^\times$) and permutation of the set of generators x, y, z of F_2 (subject to the relation $xyz = 1$), where x, y, z record the local monodromies over $0, 1, \infty$, respectively.

One of the elements x, y, z must map to a generator (i.e. nonzero element) of $\mathbb{Z}/5\mathbb{Z}$, and so by permuting x, y, z if necessary, we assume that $\mu(x) \neq 0$. Using the automorphism group of $\mathbb{Z}/5\mathbb{Z}$, we can further assume $\mu(x) = 1$. Then each of the five possible values for $\mu(y) \in \mathbb{Z}/5\mathbb{Z}$ yields a cover; the value of $\mu(z)$ is then determined by the condition $\mu(z) = -\mu(x) - \mu(y)$. The triples we obtain are displayed in the table below.

$\mu(x)$	$\mu(y)$	$\mu(y)$
1	0	4
1	1	3
1	2	2
1	3	1
1	4	0

Now note that there are only two isomorphism classes of triples on this list! The first and last are equivalent by exchanging y and z. Meanwhile, the tuple $(1, 2, 2)$ is equivalent to the tuple $(3, 1, 1)$, under the automorphism that multiplies by 3, and the symmetry among x, y, z allows us to realize all three of the middle entries as belonging to the same isomorphism class.

We saw in the case $g = 2$ above that setting $\sigma_0 = \sigma_1$ the same 5-cycle produces a $\mathbb{Z}/5\mathbb{Z}$ cover with total space of genus 2. This can be realized explicitly as the projection onto the y-coordinate of the curve $y(y-1) = x^5$. For the case of $g = 0$, set σ_0 to be a 5-cycle and σ_1 trivial. This gives a $\mathbb{Z}/5\mathbb{Z}$ cover $\widehat{\mathbb{C}}$ ramified only over $0, \infty$, and we see that this can be realized explicitly by the map $z \mapsto z^5$.

(4) Find a Belyi polynomial $p(z)$ of degree 5 such that $p^{-1}([0,1])$ is homeomorphic to the letter Y, with the fork at $z = 0$. (The Belyi condition means that $p(0) = 0, p(1) = 1$, and the critical values of p are contained in $\{0, 1\}$.)

Solution: Since the degree is 5, there must be 5 edges in the tree T. There are two distinct "Y-trees": one with branches (emanating from the "fork" (the unique vertex of degree 3)) of lengths 1,1,3, and the other with branches of lengths 1,2,2. We will consider the first of these. Along the branch of degree 3, the central vertex must go to 0. We choose to position the next point at 1; this has valence 2 and hence p has a simple branched point there. The third vertex in line also has valence 2 and will map to 0, but its position t in the complex plane is yet unspecified (though $t \neq 0, 1$). The remaining vertices are all unbranched and will not feature in our argument.

From what we have done, we have found that $p'(z)$ can be specified as follows:

$$p'(z) = az^2(z-1)(z-t).$$

We integrate this to find that

$$p(z) = \frac{a}{5}z^5 + \frac{-a(1+t)}{4}z^4 + \frac{at}{3}z^3 + c.$$

As usual, the condition $p(0) = 0$ sets $c = 0$. We have two additional equations: $p(1) = 1$ and $p(t) = 0$. These become

$$1 = \frac{a}{5} - \frac{a(t+1)}{4} + \frac{at}{3}$$

and

$$0 = \frac{a}{5}t^5 + \frac{-a(1+t)}{4}t^4 + \frac{at}{3}t^3.$$

Since $a, t \neq 0$, this second simplifies:

$$0 = \frac{t}{5} + \frac{-(1+t)}{4} + \frac{1}{3},$$

giving $t = \frac{5}{3}$. This then plugs into the first equation to yield $a = \frac{45}{4}$.