Note: throughout, X denotes a compact Riemann surface.

(1) (a) For which values of g does there exist a compact Riemann surface of genus g admitting a degree 5 map $f : X \to \widehat{\mathbb{C}}$ with $B(f) = \{0, 1, \infty\}$? [Note: there may be more than one topological type of branched cover f for a given genus. I'm not asking you to enumerate them all, just to tell me about the possible genera.]

(b) For each g which exists, choose one of the possible topological types and give an explicit pair of elements $\sigma_0, \sigma_1 \in S_5$ describing the monodromy of X over 0 and 1. (In other words, describe the associated map $\rho : \pi_1(\widehat{\mathbb{C}} - B(f)) \to S_5$ by giving its values on a loop around 0 and a loop around 1).

(c) Enumerate all such covers that are Galois (i.e. regular in the sense of covering space theory, equivalently that the field extension $\mathcal{M}(X)/\mathbb{C}(x)$ is Galois). For bonus credit, give explicit models of each possible type.

(2) Let $\omega \in \Omega(X)$ be a holomorphic 1-form. Prove that for any $p \in X$, there is a coordinate z on X near p in which $\omega = z^n \, dz$ for some $n \geq 0$.

(3) (Harmonic functions and forms)

(a) Let $u : X \to \mathbb{C}$ be a smooth complex-valued function. Prove that the following conditions are equivalent; such u is called a *harmonic function.*

(i) $\Delta u = d^2u/dx^2 + d^2u/dy^2 = 0$ in local coordinates $z = x + iy$,

(ii) $\overline{\partial}\partial u = 0$,

(iii) ∂u is a holomorphic 1-form,

(iv) $\overline{\partial} u$ is an anti-holomorphic 1-form,

(v) Locally, $u = f + \overline{g}$ where f, g are holomorphic.

(b) Now let α be a 1-form on X. Prove that the following conditions are equivalent; such α is called a *harmonic 1-form*.

(i) Locally $\alpha = du$ with u harmonic,

(ii) $\partial\alpha = \overline{\partial}\alpha = 0$,

(iii) There exist $\omega_1, \omega_2 \in \Omega(X)$ such that $\alpha = \omega_1 + \overline{\omega_2}$.

(4) (a) Let $\Lambda \subset \mathbb{C}$ be a lattice, and let $X = \mathbb{C}/\Lambda$ be the associated compact Riemann surface of genus 1. Determine $\dim(\Omega(X))$, and find an explicit basis of holomorphic differential 1-forms. [Hint: construct a Λ-invariant form on $\mathbb{C}$; this is not hard.]

(b) For each basis element as constructed in (a), determine the image of the period mapping $\int \omega : H_1(X; \mathbb{Z}) \to \mathbb{C}$.

(c) Let $\Lambda_1, \Lambda_2 \subset \mathbb{C}$ be lattices. Prove that the compact Riemann surfaces $X_1 = \mathbb{C}/\Lambda_1$ and $X_2 = \mathbb{C}/\Lambda_2$ are isomorphic (as Riemann surfaces) if and only if there is $\lambda \in \mathbb{C}^*$ such that $\Lambda_2 = \lambda\Lambda_1$ as subsets of $\mathbb{C}$. [Hint: If $f : X_1 \to X_2$ is holomorphic, how are the periods of $\omega \in \Omega(X_2)$ and $f^*(\omega) \in \Omega(X_1)$ related?]