Note: throughout, X denotes a compact Riemann surface.

(1) Prove that the local coordinates definition we gave of $\overline{\partial}$, namely

$$\overline{\partial} f = \frac{\partial f}{\partial \overline{z}}(z) d\overline{z},$$

is well-defined on a Riemann surface, i.e. independent of the choice of local holomorphic coordinate z.

(2) Let $\omega \in \Omega(X)$ be a holomorphic 1-form. Prove that for any $p \in X$, there is a coordinate z on X near p in which $\omega = z^n \, dz$ for some $n \geq 0$.

(3) (Harmonic functions and forms)
   (a) Let $u : X \to \mathbb{C}$ be a smooth complex-valued function. Prove that the following conditions are equivalent; such u is called a *harmonic function.*
      (i) $\Delta u = d^2u/dx^2 + d^2u/dy^2 = 0$ in local coordinates $z = x + iy$,
      (ii) $\overline{\partial}\partial u = 0$,
      (iii) ∂u is a holomorphic 1-form,
      (iv) $\overline{\partial} u$ is an anti-holomorphic 1-form,
      (v) Locally, $u = f + \overline{g}$ where f, g are holomorphic.
   (b) Now let α be a 1-form on X. Prove that the following conditions are equivalent; such α is called a *harmonic 1-form*.
      (i) Locally $\alpha = du$ with u harmonic,
      (ii) $\partial \alpha = \overline{\partial} \alpha = 0$,
      (iii) There exist $\omega_1, \omega_2 \in \Omega(X)$ such that $\alpha = \omega_1 + \overline{\omega_2}$.

(4) Let $X = \mathbb{CP}^1$, with local coordinate z in one chart and $w = 1/z$ in the other chart.
   (a) Let ω be a meromorphic 1-form on $\mathbb{CP}^1$. Show that if $\omega = f(z)dz$ in the coordinate z, then f must be a rational function of z.
   (b) Show that conversely, if $f(z)$ is a rational function, then the expression $f(z)dz$ extends to define a meromorphic 1-form on $\mathbb{CP}^1$.
   (c) Show that there are no nonzero holomorphic 1-forms on $\mathbb{CP}^1$.
   (d) Find all zeroes and poles (with orders and residues, where appropriate) of the 1-form defined by dz.
   (e) Do the same for dz/z.

(5) (a) Let $\Lambda \subset \mathbb{C}$ be a lattice, and let $X = \mathbb{C}/\Lambda$ be the associated compact Riemann surface of genus 1. Find an explicit basis for $\Omega(X)$, the vector space of holomorphic differential 1-forms on X. Determine $\dim(\Omega(X))$. [Hint: construct a Λ-invariant form on $\mathbb{C}$; this is not hard.]
   (b) For each basis element as constructed in (a), determine the image of the period mapping $\int \omega : H_1(X; \mathbb{Z}) \to \mathbb{C}$.
   (c) Let $\Lambda_1, \Lambda_2 \subset \mathbb{C}$ be lattices. Prove that the compact Riemann surfaces $X_1 = \mathbb{C}/\Lambda_1$ and $X_2 = \mathbb{C}/\Lambda_2$ are isomorphic (as Riemann surfaces) if and only if there is

$\lambda \in \mathbb{C}^*$ such that $\Lambda_2 = \lambda\Lambda_1$ as subsets of $\mathbb{C}$. [Hint: If $f : X_1 \to X_2$ is holomorphic, how are the periods of $\omega \in \Omega(X_2)$ and $f^*(\omega) \in \Omega(X_1)$ related?]