Note: throughout, X denotes a compact Riemann surface.

(1) Prove that the local coordinates definition we gave of $\overline{\partial}$, namely

$$\overline{\partial} f = \frac{\partial f}{\partial \overline{z}}(z) d\overline{z},$$

is well-defined on a Riemann surface, i.e. independent of the choice of local holomorphic coordinate z.

Solution: Suppose w is another choice of local coordinate, with $z = z(w)$ holomorphic. Constructing $\overline{\partial} f$ in z-coordinates and then applying the change of coordinates formula from z to w yields formally

$$\begin{aligned}\overline{\partial} f &= \frac{\partial f}{\partial \overline{z}}(z(w)) d(\overline{z}(w)) \\ &= \frac{\partial f}{\partial \overline{z}}(z(w)) \left(\frac{\partial \overline{z}}{\partial w}(w) dw + \frac{\partial \overline{z}}{\partial \overline{w}}(w) d\overline{w} \right).\end{aligned}$$

Since z is a holomorphic function of w, also $\overline{z}$ is an *anti*-holomorphic function of w, so that the first term $\frac{\partial \overline{z}}{\partial w} = 0$. Then applying the chain rule,

$$\begin{aligned}\frac{\partial f}{\partial \overline{z}}(z(w)) \left(\frac{\partial \overline{z}}{\partial w}(w) dw + \frac{\partial \overline{z}}{\partial \overline{w}}(w) d\overline{w} \right) &= \frac{\partial f}{\partial \overline{z}}(z(w)) \frac{\partial \overline{z}}{\partial \overline{w}}(w) d\overline{w} \\ &= \frac{\partial f}{\partial \overline{w}}(w) d\overline{w},\end{aligned}$$

which is the expression used to define $\overline{\partial} f$ locally in w-coordinates, as desired.

(2) Let $\omega \in \Omega(X)$ be a holomorphic 1-form. Prove that for any $p \in X$, there is a coordinate z on X near p in which $\omega = z^n \, dz$ for some $n \geq 0$.

Solution: Choose a local coordinate w near p, so that $\omega = f(w)dw$; normalize so that p is sent to 0. There is then a unique holomorphic function $g(w)$ such that $g(0) \neq 0$ and such that $f(w) = w^n g(w)$ for some $n \geq 0$. Let's try and reverse-engineer the properties of the desired coordinate z: what properties must $z = z(w)$ satisfy if $\omega = z^n dz$ in z-coordinates?

Writing $z = z(w)$, we compute the pullback, and compare to what we already have:

$$\omega = z(w)^n dz = z(w)^n \frac{dz}{dw} dw = w^n g(w) dw.$$

Comparing coefficients, we obtain the separable differential equation

$$z^n \, dz = w^n g(w) \, dw = f(w) dw$$

Solving by integrating, we find that $z = \left((n+1) \int f(w) \, dw \right)^{1/n+1}$ is the desired expression. To check that it is a *coordinate*, it suffices by the inverse function theorem to check that $dz/dw(0) \neq 0$. We obtained an expression for dz/dw above:

$$\frac{dz}{dw} = \frac{w^n}{z^n} g(w).$$

Since f vanishes to order n at $w = 0$, $\int f(w)dw$ vanishes to order $n+1$, so that z vanishes to order 1 at $w = 0$. We conclude that w^n/z^n is nonzero, and since $g(0) \neq 0$ by construction as well, we conclude that $dz/dw \neq 0$ as desired.

A comment: it is no accident that where ω is nonvanishing, the coordinate is given by integrating ω itself. After all, a function $f : U \to \mathbb{C}$ with local coordinate w determines a coordinate $z = f(w)$ near $p \in U$ if and only if $f_w \neq 0$. In the situation where ω is nonvanishing at p, integrating $\omega = f(w)dw$ therefore gives a local coordinate z near p; in these coordinates, dz on the codomain pulls back to $d\int f(w)dw = f(w)dw = \omega$ as desired!

(3) (Harmonic functions and forms)
  (a) Let $u : X \to \mathbb{C}$ be a smooth complex-valued function. Prove that the following conditions are equivalent; such u is called a *harmonic function.*
    (i) $\Delta u = d^2u/dx^2 + d^2u/dy^2 = 0$ in local coordinates $z = x + iy$,
    (ii) $\overline{\partial}\partial u = 0$,
    (iii) ∂u is a holomorphic 1-form,
    (iv) $\overline{\partial} u$ is an anti-holomorphic 1-form,
    (v) Locally, $u = f + \overline{g}$ where f, g are holomorphic.
  (b) Now let α be a 1-form on X. Prove that the following conditions are equivalent; such α is called a *harmonic 1-form*.
    (i) Locally $\alpha = du$ with u harmonic,
    (ii) $\partial\alpha = \overline{\partial}\alpha = 0$,
    (iii) There exist $\omega_1, \omega_2 \in \Omega(X)$ such that $\alpha = \omega_1 + \overline{\omega_2}$.

Solution: We begin with (a). To show the equivalence of (i) and (ii), we expand:

$$\overline{\partial}\partial u = \frac{\partial^2 u}{\partial\overline{z}\partial z} d\overline{z}dz$$

As $\partial/\partial z = \partial/\partial x - i\partial/\partial y$ and $\partial/\partial\overline{z} = \partial/\partial x + i\partial/\partial y$, we compose these to see

$$\frac{\partial^2 u}{\partial\overline{z}\partial z} = \frac{\partial}{\partial\overline{z}}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 u}{\partial y\partial x} - i\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} = \Delta u,$$

with the last equality holding by commutativity of mixed partials. This shows the equivalence of (i) and (ii).

Supposing $\overline{\partial}\partial u = 0$, we see that $\partial u = \partial u/\partial z\, dz$ is of type $(1,0)$ by construction, and then the condition $\overline{\partial}\partial u = 0$ just asserts that ∂u is holomorphic. The converse holds by retracing the argument backwards. Since $\overline{\partial}\partial = -\partial\overline{\partial}$ (the minus sign coming from the fact that $d\overline{z}dz = -dzd\overline{z}$), repeating this same argument then shows the equivalence of (ii) with (iv). Finally, since ∂u is holomorphic, it has a local expression via $\partial u = df$ for f holomorphic, and likewise $\overline{\partial} u = d\overline{g}$, again for g holomorphic. Then $du = (\partial + \overline{\partial})(f) = d(f + \overline{g})$, so $u = f + \overline{g}$ up to a constant. Conversely, any $u = f + \overline{g}$ is easily seen to satisfy $\overline{\partial}\partial u = 0$.

For (b), suppose α admits a local expression $\alpha = du$ with u harmonic. Then

$$\partial\alpha = \partial(\partial + \overline{\partial})u = \partial\overline{\partial}u = 0,$$

and an analogous calculation shows $\overline{\partial}\alpha = 0$. Next, suppose $\partial\alpha = \overline{\partial}\alpha = 0$. Decompose α into its $(1,0)$ and $(0,1)$-parts; then locally

$$\alpha = f(z)dz + g(z)d\overline{z},$$

with $\alpha_{(1,0)}$ represented locally as $f(z)dz$ and $\alpha_{(0,1)}$ represented as $g(z)d\overline{z}$. The condition $\partial\alpha = 0$ reduces to $\partial g(z) = 0$, showing g is antiholomorphic; likewise $\overline{\partial}\alpha$ implies that f is holomorphic, showing the decomposition $\alpha = \omega_1 + \overline{\omega_2}$. Finally, any $\omega_1 + \overline{\omega_2}$ is easily seen to be annihilated by both ∂ and $\overline{\partial}$.

(4) Let $X = \mathbb{CP}^1$, with local coordinate z in one chart and $w = 1/z$ in the other chart.
   (a) Let ω be a meromorphic 1-form on $\mathbb{CP}^1$. Show that if $\omega = f(z)dz$ in the coordinate z, then f must be a rational function of z.
   (b) Show that conversely, if $f(z)$ is a rational function, then the expression $f(z)dz$ extends to define a meromorphic 1-form on $\mathbb{CP}^1$.
   (c) Show that there are no nonzero holomorphic 1-forms on $\mathbb{CP}^1$.
   (d) Find all zeroes and poles (with orders and residues, where appropriate) of the 1-form defined by dz.
   (e) Do the same for dz/z.

Solution: (a) Since ω is assumed to be meromorphic, necessarily $f(z)$ is meromorphic, and the expression for ω in w-coordinates must likewise be meromorphic. Computing, this is given on the overlap $w \neq 0$ as

$$g(w)dw = f(w^{-1})\tfrac{dz}{dw}dw = -w^{-2}f(w^{-1})dw.$$

The zeroes and poles of a meromorphic function are isolated, so that there is a neighborhood $0 < |w| < \varepsilon$ on which $g(w)$ is holomorphic and nonvanishing. It follows that $f(z)$ is holomorphic and nonvanishing for $|z| > 1/\varepsilon$, and hence f has only finitely many zeroes and poles. Thus f admits an expression of the form

$$f(z) = p(z)/q(z),$$

where $q(z)$ is a polynomial and $p(z)$ is an entire function. Converting to w-coordinates once again, on $w \neq 0$,

$$g(w) = w^{-2}\frac{p(w^{-1})}{q(w^{-1})}.$$

Supposing $\deg(q) = d$, then $q(w^{-1}) = w^{-d}r(d)$ for some *polynomial* $r(d)$ (in fact r is just obtained from q by reversing the coefficients; in particular, $r(0) \neq 0$). Inserting,

$$g(w) = w^{d-2}\frac{p(w^{-1})}{r(w)}.$$

By definition, since $g(w)$ is meromorphic at $w = 0$, there is some integer m such that the limit

$$\lim_{w\to 0} w^{d-2+m}\frac{p(w^{-1})}{r(w)}$$

exists and is nonzero. Now consider the (convergent!) Taylor series expansion for $p(z)$ at $z = 0$, say $p(z) = \sum_{k \geq 0} a_k z^k$, and convert to a Laurent series expression $p(w^{-1}) = \sum_{k \geq 0} a_k w^{-k}$. Inserting this into the above limit, we conclude that $a_k = 0$ for k sufficiently large, i.e. that p is also a polynomial, and hence $f(z) = p(z)/q(z)$ is rational as claimed.

(b) This is easy: if $f(z) = p(z)/q(z)$ is rational, then the expression for $\omega = f(z)dz$ in w-coordinates is $w^{-2}p(w^{-1})/q(w^{-1})dw$. As we saw in the previous step, both of $p(w^{-1})$ and $q(w^{-1})$ are polynomials in w up to a factor of w, so that ω transparently admits an expression as a rational function of w as well, and hence is meromorphic on all of $\mathbb{CP}^1$.

(c) If ω is globally holomorphic, then since in z coordinates it is represented as $\omega = f(z)d(z)$ with f rational, then in fact $f(z)$ must be a polynomial, say of degree $d \geq 0$. But then, as in (a), in w coordinates it has the expression $\omega = w^{-2-d}g(w)dw$ for some polynomial $g(w)$ *not vanishing at* $w = 0$ and so necessarily has a pole of order ≥ 2 at $w = 0$.

(d) This was already addressed in the previous problem: $\omega = dz$ has no zeroes and single double pole at $w = 0$. This pole has no residue, since the local expression $\omega = \frac{1}{w^2}dw$ has vanishing $\frac{1}{w}$ term.

(e) The form $\omega = \frac{dz}{z}$ has a simple pole of residue 1 at $z = 0$. In w coordinates, this transforms to $\frac{-dw}{w}$, which has a simple pole of opposite residue -1 at $w = 0$. Note that the sums of the orders (-2) and residues (0) are the same in both examples, as we know from class they must be for any pair of meromorphic differentials on the same compact Riemann surface.

(5) (a) Let $\Lambda \subset \mathbb{C}$ be a lattice, and let $X = \mathbb{C}/\Lambda$ be the associated compact Riemann surface of genus 1. Find an explicit basis for $\Omega(X)$, the vector space of holomorphic differential 1-forms on X. Determine $\dim(\Omega(X))$. [Hint: construct a Λ-invariant form on $\mathbb{C}$; this is not hard.]
   (b) For each basis element as constructed in (a), determine the image of the period mapping $\int \omega : H_1(X; \mathbb{Z}) \to \mathbb{C}$.
   (c) Let $\Lambda_1, \Lambda_2 \subset \mathbb{C}$ be lattices. Prove that the compact Riemann surfaces $X_1 = \mathbb{C}/\Lambda_1$ and $X_2 = \mathbb{C}/\Lambda_2$ are isomorphic (as Riemann surfaces) if and only if there is $\lambda \in \mathbb{C}^*$ such that $\Lambda_2 = \lambda\Lambda_1$ as subsets of $\mathbb{C}$. [Hint: If $f : X_1 \to X_2$ is holomorphic, how are the periods of $\omega \in \Omega(X_2)$ and $f^*(\omega) \in \Omega(X_1)$ related?]

Solution: (a) From the injectivity of the period map we established in class, we see that $\dim(\Omega(X)) \leqslant g = 1$. Thus, we need only to produce some nonzero holomorphic form. Following the suggestion, we observe that $dz \in \Omega(\mathbb{C})$ is invariant under the transformation $z \mapsto z + c$ for any $c \in \mathbb{C}$; in particular, for $z \in \Lambda$. Thus dz descends to give a nonzero form on $\Omega(X)$.

(b) From (a), we must determine the periods of $dz \in \Omega(\mathbb{C}/\Lambda)$. Choose a pair of generators $z_1, z_2 \in \Lambda$, and observe that $H_1(X; \mathbb{Z})$ is generated by the images of segments from 0 to z_1 and z_2 in $\mathbb{C}$. By basic calculus, $\int_0^{z_i} dz = z_i$, showing that the image of $H_1(X; \mathbb{Z})$ in $\mathbb{C}$ under the period mapping is just the lattice Λ!

(c) First we note that the "if" direction is trivial: if $\Lambda_2 = \lambda\Lambda_1$, then $z \mapsto \lambda z$ descends from $\mathbb{C}$ to give an isomorphism $\mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$. For the "only if" direction, suppose that there is an isomorphism $f : X_1 \to X_2$. Using the result of (a), let $dz \in \Omega(X_2)$ be given Note that

$f^*(dz) \in \Omega(X_1)$ is necessarily nonzero, as can be seen in local coordinates. Invoking (a) again, $f^*(dz) = \lambda dz \in \Omega(X_1)$.

To proceed, let us consider the hint. Using the theory of integration of differential forms, we find that for any $[\gamma] \in H_1(X_1; \mathbb{Z})$,

$$\int_\gamma f^*(\omega) = \int_{f_*(\gamma)} \omega.$$

We conclude: *any period of* $f^*(\omega)$ *is necessarily a period of* ω. Applying this to $f^*(dz)$ as in our setting, we find that any period of $f^*(dz) \in \Omega(X_1)$ is necessarily a period of $dz \in \Omega(X_2)$. By (b), this means that any period of $f^*(dz)$ is contained in Λ_2. But since $f^*(dz) = \lambda dz$, we conclude that $\lambda \Lambda_1 \subset \Lambda_2$. Applying the same argument to the inverse map f^{-1}, we conclude that this containment is an equality, as desired.