Note: this set might look longer than usual, but in fact I've taken a set of standard length and divided it up into finer pieces to help you along the way.

(1) Let $\omega \in \Omega(X)$ be a holomorphic 1-form on a compact Riemann surface X, and let

$$\Lambda = \left\{ \int_\gamma \omega \mid \gamma \in \pi_1(X) \right\} \subset \mathbb{C}$$

be the group of periods of ω.

(a) Show that Λ cannot be isomorphic to $\mathbb{Z}$. [Hint: supposing this were the case, construct a nonconstant holomorphic function on X.]

(b) Show that if Λ is a lattice in $\mathbb{C}$, then there exists a nonconstant holomorphic map f from X to a Riemann surface of genus 1.

(2) (Fermat curves) Let X_d be the compact Riemann surface defined as the solution to $x^d + y^d = z^d$ in $\mathbb{CP}^2$. Such X_d is called the *degree-d Fermat curve*.

(a) Let $X_d^\circ \subset \mathbb{C}^2$ be the portion of X lying in the affine plane $\mathbb{C}^2 \subset \mathbb{CP}^2$ defined by $z = 1$. Write down an equation $f(x, y) = 0$ defining X°. Use this to find the branch locus $B \subset X^\circ$, with multiplicities, for the projection of X° onto the x-coordinate.

(b) Repeat the above process, this time exchanging the roles of the x and z coordinate.

(c) Are any of the branch points you find in (a) distinct from the branch points you find in (b)? Combine your analysis to completely describe the branch locus of the meromorphic function $\pi : X_d \to \mathbb{CP}^1$ given by $\pi([x : y : z]) = [x : z]$.

(d) Use the previous steps and Riemann-Hurwitz to show that X_d has genus $\binom{d-1}{2}$.

(3) Let $X = X_4$ be the Fermat curve of degree 4; by the previous problem X_4 has genus 3. In this problem you will construct a basis for $\Omega(X)$. As in the previous problem, let X° be the portion of X lying in the affine plane determined by $z = 1$, with equation $x^4 + y^4 = 1$.

(a) Let $\pi : X^\circ \to \mathbb{C}, (x, y) \mapsto x$ be the projection onto the x-coordinate. Show that $\pi^*(dx)$ is holomorphic on X° and has a zero of order 3 at each of the branch points of π. [Hint: recall the result we proved in the first week that a branch cover is locally equivalent to $z \mapsto z^{d_z}$.]

(b) Show that $\omega_1 = \pi^*(dx)/y^3$ is holomorphic and has no zeroes on X°.

(c) Let $\alpha : X \to X$ be the symmetry given by $[x : y : z] \mapsto [ix : y : z]$. Show that α acts transitively on the points $X \setminus X^\circ$, and that ω_1 is an eigenvector for $\alpha^* : \Omega(X^\circ) \to \Omega(X^\circ)$.

(d) Use the previous step to argue that ω_1 admits a holomorphic extension to X, with a simple zero at each point of $X \setminus X^\circ$.

(e) Show that $\omega_2 = x\omega_1$ and $\omega_3 = y\omega_1$ in $\Omega(X^\circ)$ likewise extend to holomorphic forms on X.

(f) Show that $\{\omega_1, \omega_2, \omega_3\}$ forms a basis for $\Omega(X)$. [Hint: the functions $1, x, y$ are linearly independent on X].

(4) Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves on a topological space X, say induced by maps of sheaves $\alpha : \mathcal{A} \to \mathcal{B}$ and $\beta : \mathcal{B} \to \mathcal{C}$. Consider the commutative diagram below.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{A}(X) & \longrightarrow & \mathcal{B}(X) & \longrightarrow & \mathcal{C}(X) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \prod \mathcal{A}_p & \longrightarrow & \prod \mathcal{B}_p & \longrightarrow & \prod \mathcal{C}_p. \end{array}$$

(a) Verify that the bottom row of the diagram is exact and that the vertical maps are injective. [This is not hard.]
(b) Building off of these facts and what you know about sheaves, prove that the top row is exact.

(5) Let $\phi : \mathcal{A} \to \mathcal{B}$ be a morphism of sheaves.
(a) Formulate definitions of the presheaves $\operatorname{Ker} \phi, \operatorname{Im} \phi$, and $\operatorname{Coker} \phi$.
(b) Prove that $\operatorname{Ker} \phi$ is in fact a sheaf.
(c) By considering the morphism $\exp : \mathcal{O} \to \mathcal{O}^*$, show that neither $\operatorname{Im} \phi$ nor $\operatorname{Coker} \phi$ is in general a sheaf.