(1) Let $\omega \in \Omega(X)$ be a holomorphic 1-form on a compact Riemann surface X, and let

$$\Lambda = \left\{ \int_\gamma \omega \mid \gamma \in \pi_1(X) \right\} \subset \mathbb{C}$$

be the group of periods of ω.

(a) Show that Λ cannot be isomorphic to $\mathbb{Z}$. [Hint: supposing this were the case, construct a nonconstant holomorphic function on X.]

(b) Show that if Λ is a lattice in $\mathbb{C}$, then there exists a nonconstant holomorphic map f from X to a Riemann surface of genus 1.

Solution: For (a), suppose that $\Lambda = \langle \lambda \rangle$ were isomorphic to $\mathbb{Z}$. Define the function $f : X \to \mathbb{C}$ by

$$f(x) = \exp\left(\frac{2\pi i}{\lambda} \int_{x_0}^{x} \omega \right),$$

where $x_0 \in X$ is arbitrary and the integral is taken along an arbitrary path. Since every period of ω is a multiple of λ by hypothesis, f is a globally well-defined nonconstant holomorphic function. But X is compact, and so such functions do not exist.

For (b), the function

$$g(x) = \int_{x_0}^{x} \omega \pmod{\Lambda}$$

gives a globally well-defined holomorphic map $g : X \to \mathbb{C}/\Lambda$.

(2) (Fermat curves) Let X_d be the compact Riemann surface defined as the solution to $x^d + y^d = z^d$ in $\mathbb{CP}^2$. Such X_d is called the *degree-d Fermat curve*.

(a) Let $X_d^\circ \subset \mathbb{C}^2$ be the portion of X lying in the affine plane $\mathbb{C}^2 \subset \mathbb{CP}^2$ defined by $z = 1$. Write down an equation $f(x, y) = 0$ defining X°. Use this to find the branch locus $B \subset X^\circ$, with multiplicities, for the projection of X° onto the x-coordinate.

(b) Repeat the above process, this time exchanging the roles of the x and z coordinate.

(c) Are any of the branch points you find in (a) distinct from the branch points you find in (b)? Combine your analysis to completely describe the branch locus of the meromorphic function $\pi : X_d \to \mathbb{CP}^1$ given by $\pi([x : y : z]) = [x : z]$.

(d) Use the previous steps and Riemann-Hurwitz to show that X_d has genus $\binom{d-1}{2}$.

Solution: For (a), we obtain the equation for X° simply by setting $z = 1$, giving $x^d + y^d = 1$. We see that the projection $(x, y) \to x$ has degree d. Thus the branch locus consists of points where there are fewer than d points (x, y) on X° with given x-coordinate. This is clearly the set of points where $y = 0$, i.e. where $x^d = 1$. Since we drop from d points to 1 over such points, the multiplicity is d at each point.

For (b), we find that the equation is now $1 + y^d = z^d$. The branch locus for the projection onto the z-coordinate again consists of points where $y = 0$, again given as the solution to $z^d = 1$, and again with multiplicity d.

For (c), these are actually all the same points. Since we set $z = 1$ in (a) and found the affine coordinates to be $(\zeta^k, 0)$ for ζ a primitive d^{th} root of unity and $0 \leqslant k \leqslant d-1$, the projective coordinates are $[\zeta^k : 0 : 1]$. In (b), we set $x = 1$ and found affine coordinates $(y, z) = (0, \zeta^k)$, corresponding to the projective coordinates $[1 : 0 : \zeta^k]$. As $[1 : 0 : \zeta^k] = [\zeta^{-k} : 0 : 1]$, we see that these points are in fact the same. Thus there is a grand total of d branch points, each of multiplicity d.

For (d), we found the degree of the projection to be d, and that there are d branch points, each of multiplicity d. Applying Riemann-Hurwitz, we see

$$2 - 2g(X_d) = d \cdot 2 - d \cdot (d-1).$$

Solving, we find $g(X_d) = (d-1)(d-2)/2 = \binom{d-1}{2}$.

(3) Let $X = X_4$ be the Fermat curve of degree 4; by the previous problem X_4 has genus 3. In this problem you will construct a basis for $\Omega(X)$. As in the previous problem, let X° be the portion of X lying in the affine plane determined by $z = 1$, with equation $x^4 + y^4 = 1$.

   (a) Let $\pi : X^\circ \to \mathbb{C}, (x, y) \mapsto x$ be the projection onto the x-coordinate. Show that $\pi^*(dx)$ is holomorphic on X° and has a zero of order 3 at each of the branch points of π. [Hint: recall the result we proved in the first week that a branch cover is locally equivalent to $z \mapsto z^{d_z}$.]

   (b) Show that $\omega_1 = \pi^*(dx)/y^3$ is holomorphic and has no zeroes on X°.

   (c) Let $\alpha : X \to X$ be the symmetry given by $[x : y : z] \mapsto [ix : y : z]$. Show that α acts transitively on the points $X \setminus X^\circ$, and that ω_1 is an eigenvector for $\alpha^* : \Omega(X^\circ) \to \Omega(X^\circ)$.

   (d) Use the previous step to argue that ω_1 admits a holomorphic extension to X, with a simple zero at each point of $X \setminus X^\circ$.

   (e) Show that $\omega_2 = x\omega_1$ and $\omega_3 = y\omega_1$ in $\Omega(X^\circ)$ likewise extend to holomorphic forms on X.

   (f) Show that $\{\omega_1, \omega_2, \omega_3\}$ forms a basis for $\Omega(X)$. [Hint: the functions $1, x, y$ are linearly independent on X].

Solution: For (a), certainly dx itself is holomorphic on $\mathbb{C}$, and the pullback of a holomorphic form along a holomorphic map remains holomorphic. From the previous problem, the multiplicity of each branch point is $d = 4$, and so by the result from class, near each branch point, there is a coordinate z on X° in which π is given by $x = z^4$. Pulling back dx in these coordinates, we find $\pi^*(dx) = d(z^4) = 4z^3 dz$ has a zero of order 3.

For (b), the function $1/y^3$ has a triple pole along the locus $y = 0$. By the analysis of the previous problem, the branch points of π occur precisely where $y = 0$. Thus the triple pole of $1/y^3$ cancels out the triple zero of $\pi^*(dx)$, yielding a form that is holomorphic and nonvanishing on X°.

(c) is a direct computation: the points $X \setminus X^\circ$ are the points on X for which $z = 0$, which are easily seen to be the four points $[1 : \zeta : 0]$, where ζ is an 8^{th} root of unity that is not a 4^{th} root of unity. These are permuted transitively by α. Under α, dx pulls back to $d(ix) = idx$ and y remains y, so that $\alpha^*(\omega_1) = i\omega_1$.

For (d), we know that ω_1 is holomorphic and nonvanishing on X°. Since the total degree of any meromorphic form is $2g - 2 = 4$, we know that there must be a total of four zeroes hiding at the points $X \setminus X^\circ$. By itself this is not enough to guarantee that these arise as a simple zero at each point - *a priori* there could be a pole at some point and a double zero elsewhere. But from (c) we know that the points of $X \setminus X^\circ$ are transitively permuted by α, and that ω_1 is an eigenvector, so that the order of vanishing at each point of $X \setminus X^\circ$ must be equal, hence a simple zero.

(e) is an elaboration of the same ideas. x has a simple pole at infinity. Multiplying this to make $\omega_2 = x\omega_1$ therefore cancels out the zeroes at $X \setminus X^\circ$, replacing them with four simple zeroes where $x = 0$; this is still holomorphic. Likewise y has a simple pole at infinity, so that $\omega_3 = y\omega_1$ similarly remains holomorphic.

(f) By our analysis of the period map in class, we know that $\dim(\Omega(X)) \leqslant 3$, so it suffices to show that $\{\omega_1, \omega_2, \omega_3\}$ is linearly independent. If there are scalars $a, b, c \in \mathbb{C}$ such that $a\omega_1 + b\omega_2 + c\omega_3 = 0$, then working in any local coordinate, this implies a relation $a + bx + cy$ holding across all points $(x, y) \in X^\circ$. This is clearly nonsense.

(4) Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves on a topological space X, say induced by maps of sheaves $\alpha : \mathcal{A} \to \mathcal{B}$ and $\beta : \mathcal{B} \to \mathcal{C}$. Consider the commutative diagram below.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}(X) \arrow[r] \arrow[d] & \mathcal{B}(X) \arrow[r] \arrow[d] & \mathcal{C}(X) \arrow[d] \\
0 \arrow[r] & \prod \mathcal{A}_p \arrow[r] & \prod \mathcal{B}_p \arrow[r] & \prod \mathcal{C}_p.
\end{tikzcd}$$

  (a) Verify that the bottom row of the diagram is exact and that the vertical maps are injective. [This is not hard.]
  (b) Building off of these facts and what you know about sheaves, prove that the top row is exact.

Solution: For (a), exactness along the bottom row follows from the definition of exactness of a sequence of sheaves: this holds precisely when it holds at each stalk, as encoded in the bottom row. The sheaf axioms imply that if $f \in \mathcal{A}(X)$ has zero restriction to each stalk, then $f = 0$, showing injectivity of the vertical maps.

We turn to (b). To establish exactness at $\mathcal{A}(X)$, we observe that since going down and over is injective by (a), so too must going over and then down; in particular $\mathcal{A}(X) \to \mathcal{B}(X)$ must be injective as desired. To establish exactness at $\mathcal{B}(X)$, we must show that the kernel of $\beta : \mathcal{B}(X) \to \mathcal{C}(X)$ coincides with the image of $\alpha : \mathcal{A}(X)$ in $\mathcal{B}(X)$. To that end, suppose that $g \in \mathcal{B}(X)$ satisfies $\beta(g) = 0$. It follows that the image of g in $\prod \mathcal{C}_p$ going across and down is zero, and hence so too is the image going down and across. By exactness of the bottom row, there is a *unique* $(f_p) \in \prod \mathcal{A}_p$

mapping to $(g_p) \in \prod \mathcal{B}_p$. We would like to assemble (f_p) into an element $f \in \mathcal{A}(X)$. By definition, each $f_p \in \mathcal{A}_p$ is induced from some $f_{U_p} \in \mathcal{A}(U_p)$ for some open set U_p containing p. We claim that the local sections $\{f_{U_p} \in \mathcal{A}(U_p)\}$ assemble into a global section $f \in \mathcal{A}(X)$, i.e. that the restrictions of f_{U_p} and f_{U_q} to $\mathcal{A}(U_{pq})$ agree, defining $U_{pq} = U_p \cap U_q$ for convenience. By the exactness at $\mathcal{A}$ already established, as applied to U_{pq}, we know that there is a unique $f_{U_{pq}} \in \mathcal{A}(U_{pq})$ whose image in $\mathcal{B}(U_{pq})$ is $g_{U_{pq}}$. But the restriction of both f_{U_p} and f_{U_q} to U_{pq} have this property, so they must coincide, as required.

(5) Let $\phi : \mathcal{A} \to \mathcal{B}$ be a morphism of sheaves.
   (a) Formulate definitions of the presheaves $\operatorname{Ker}\phi, \operatorname{Im}\phi$, and $\operatorname{Coker}\phi$.
   (b) Prove that $\operatorname{Ker}\phi$ is in fact a sheaf.
   (c) By considering the morphism exp : $\mathcal{O} \to \mathcal{O}^*$, show that neither $\operatorname{Im}\phi$ nor $\operatorname{Coker}\phi$ is in general a sheaf.

Solution: (a) Let $\bullet$ stand for one of $\operatorname{Ker}, \operatorname{Im}, \operatorname{Coker}$. Define the presheaf $\bullet\phi$ by the assignment

$$\bullet\phi(U) = \bullet(\phi_U : \mathcal{A}(U) \to \mathcal{B}(U)).$$

We turn to (b), and claim that $\operatorname{Ker}\phi$ is a sheaf. Suppose that $f \in \operatorname{Ker}\phi(U) \leqslant \mathcal{A}(U)$ restricts to the zero section on some open cover $\{U_i\}$ of U. By the sheaf axioms for $\mathcal{A}$, it follows that f is zero as an element of $\mathcal{A}(U)$, and hence as an element of $\operatorname{Ker}\phi(U)$. If $\{f_i \in \operatorname{Ker}\phi(U_i)\}$ are coherent on overlaps, the same is true for f_i when viewed as local sections of $\mathcal{A}(U_i)$. Since $\mathcal{A}$ is a sheaf, these assemble to give an element $f \in \mathcal{A}(U)$. We claim that moreover $f \in \operatorname{Ker}\phi(U)$. Examine $\phi(f) \in \mathcal{B}(U)$. Restricting to the open cover U_i, each $\phi(f)\mid_{U_i}= 0$ by hypothesis. Since $\mathcal{B}$ is a sheaf, these local sections glue together, showing that $\phi(f) = 0$ as was to be shown.

(c) The image presheaf is not always a sheaf. For example, consider the exponential map exp : $\mathcal{O} \to \mathcal{O}^*$ of sheaves on $\mathbb{C}^*$. The section $z \in \mathcal{O}^*(\mathbb{C}^*)$ admits a local logarithm, and so there is an open cover of $\mathbb{C}^*$ on which the restriction of z is in the image of exp locally, but these fail to assemble into a global logarithm. The same example shows that the cokernel presheaf is not always a sheaf. The exponential map is locally surjective and hence surjective at the level of stalks, implying that if Coker(exp) is a sheaf, it must be the zero sheaf. But as discussed before, $z \in \mathcal{O}^*(\mathbb{C}^*)$ does not have a global logarithm, so that Coker(exp : $\mathcal{O}(\mathbb{C}^*) \to \mathcal{O}^*(\mathbb{C}^*)$) is a non-trivial group. This shows the failure of the existence of gluing of local sections for the cokernel presheaf.