(1) Let $X = \widehat{\mathbb{C}}$ with the Zariski topology (U is open iff U is empty or $U = \mathbb{C} - E$ with E finite). In the latter case define

$$\mathcal{F}(U) = \{f \in \mathbb{C}(z) \mid f \text{ has no poles on } U\},$$

and set $\mathcal{F}(\emptyset) = (0)$. Show that $\mathcal{F}$ is a sheaf on $\widehat{\mathbb{C}}$.

(2) Let E be an abstract set with a finite cover, $E = \bigcup_{i=1}^{n} E_i$, such that $\bigcap E_i = \emptyset$. Show that there are functions $\rho_i : E \to \{0, 1\}$ such that $\rho_i(x) = 0$ on E_i and $\sum_i \rho_i(x) = 1$ for all x.

(3) Consider a finite covering of $X = \widehat{\mathbb{C}}$ by Zariski-open sets $U_i = X - E_i$ with E_i finite. Let $V = \bigcap U_i$ and let $\mathcal{F}(V)$ be defined as in problem (1). Construct a family of linear maps

$$R_i : \mathcal{F}(V) \to \mathbb{C}(z)$$

such that $R_i(f)$ has no poles in E_i and $f = \sum R_i(f)$ for all $f \in \mathcal{F}(V)$. [Hint: use partial fractions.]

(4) Prove that $H^1(\widehat{\mathbb{C}}; \mathcal{F}) = 0$. [Hint: set $f_i = \sum_j R_j(g_{ij})$.]

Commentary: Note the similarity between this argument and the proof we gave in class of the vanishing of higher cohomology of sheaves of smooth functions. The functions ρ_i and R_i mimic a partition of unity, with partial fractions being the engine that solves the "local-to-global" problem of extending functions onto larger open sets. Thus we see another aspect of the slogan that sheaf cohomology is measuring local-to-global obstructions; in this case, there are none!