(1) Let X be a topological space, let $\mathbf{U} = \{U_i\}$ be a cover of X, and let $\mathbf{V} = \{V_i\}$ be a refinement of $\mathbf{U}$, so that there is a function ρ from the index set of $\mathbf{V}$ to the index set of $\mathbf{U}$ so that $V_i \subseteq U_{\rho(i)}$ holds for all $V_i \in \mathbf{V}$. Let $\mathcal{F}$ be a sheaf on X. Prove that the refinement map $H^1(\mathcal{F}, \mathbf{U}) \to H^1(\mathcal{F}, \mathbf{V})$ is injective. (If you get stuck, consult Theorem 6.3 in McMullen, but try solving it on your own first).

(2) Let $X = \widehat{\mathbb{C}}$ with the Zariski topology (U is open iff U is empty or $U = \mathbb{C} - E$ with E finite). In the latter case define

$$\mathcal{F}(U) = \{f \in \mathbb{C}(z) \mid f \text{ has no poles on } U\},$$

and set $\mathcal{F}(\emptyset) = (0)$. Show that $\mathcal{F}$ is a sheaf on $\widehat{\mathbb{C}}$.

Solution: Certainly $\mathcal{F}$ defines a presheaf, since if $f \in \mathbb{C}(z)$ has no poles on U, then *a fortiori* f has no poles on any open subset $V \subset U$. Suppose that $\{U_i\}$ is an open cover of a Zariski-open $U \subset \widehat{\mathbb{C}}$, and that $(f_i \in \mathcal{F}(U_i))$ is a system of local sections of $\mathcal{F}$ with compatible overlaps. We need to show that (f_i) assembles uniquely into some $f \in \mathcal{F}(U)$. Since meromorphic functions (with no requirements on poles) form a sheaf on $\widehat{\mathbb{C}}$, we know that (f_i) assembles into some meromorphic function f on U. But since U is covered by the open sets U_i and by hypothesis f has no poles on any U_i, it follows that f has no poles on U as was to be shown.

(3) Let E be an abstract set with a finite cover, $E = \bigcup_{i=1}^n E_i$, such that $\bigcap E_i = \emptyset$. Show that there are functions $\rho_i : E \to \{0, 1\}$ such that $\rho_i(x) = 0$ on E_i and $\sum_i \rho_i(x) = 1$ for all x.

Solution: The basic idea is to define the ρ_i in sequence, being as greedy as possible at each stage. Define $F_0 := E$ and $F_i := F_{i-1} \cap E_i$ for $i = 1, \ldots, n$. Recall that the *indicator function* χ_V of a subset $V \subset U$ is the function $\chi_V : U \to \{0, 1\}$ that takes the value 1 on V and 0 elsewhere. Then define

$$\rho_i = \chi_{F_{i-1} - F_i}.$$

Note that $E_i \in (F_{i-1} - F_i)^c$, so that $\rho_i(z) = 0$ for $z \in E_i$. Note next that each $x \in E$ is in at most one of the sets $F_{i-1} - F_i$, and in fact each x is in *exactly* one such set, since $F_n = E_1 \cap \cdots \cap E_n = \emptyset$ by hypothesis.

(4) Consider a finite covering of $X = \widehat{\mathbb{C}}$ by Zariski-open sets $U_i = X - E_i$ with E_i finite. Let $V = \bigcap U_i$ and let $\mathcal{F}(V)$ be defined as in problem (1). Construct a family of linear maps

$$R_i : \mathcal{F}(V) \to \mathbb{C}(z)$$

such that $R_i(f)$ has no poles in E_i and $f = \sum R_i(f)$ for all $f \in \mathcal{F}(V)$. [Hint: use partial fractions.]

Solution: Let $f \in \mathbb{C}(z)$ be arbitrary. Let E be the set of poles of f. The method of partial fractions gives an expression

$$f = \sum_{x \in E} f_x(z),$$

with each f_x having poles only at $x \in E$. Such an expression is unique up to passing a constant function (i.e. a function with no poles at all) from one f_x to another $f_{x'}$. We normalize this by choosing $x_0 \in E$ and requiring $f_x(x_0) = 0$ for $x \neq x_0$. Since $\{U_i\}$ covers $\widehat{\mathbb{C}}$, it follows that $\bigcap E_i = \emptyset$, and so $E, \{E_i\}$ satisfy the hypotheses of the previous problem. Let ρ_i be the functions constructed there. Define

$$R_i(f) = \sum_{x \in E} \rho_i(x) f_x,$$

and note that our construction of ρ_i ensures that $R_i(f)$ satisfies the requirements of the problem.

(5) Prove that $H^1(\widehat{\mathbb{C}}; \mathcal{F}) = 0$. [Hint: set $f_i = \sum_j R_j(g_{ij})$.]

Solution: Let (g_{ij}) be a 1-cocycle; we seek to show it is a coboundary. Unpacking the definitions, this means that there is a (Zariski-) open covering U_i of $\widehat{\mathbb{C}}$, and sections $g_{ij} \in \mathcal{F}(U_{ij})$ that satisfy the cocycle condition $g_{ij} + g_{jk} + g_{ki} = 0$ (using the convention that indices are ordered and $g_{ij} = -g_{ji}$). Our goal is to construct $f_i \in \mathcal{F}(U_i)$ such that $g_{ij} = f_i - f_j$.

Following the hint, we define

$$f_i = \sum_j R_j(g_{ij});$$

we note this is sensible since $g_{ij} \in \mathcal{F}(U_{ij})$ and hence admits a restriction to $V = \bigcap U_i$.

We first show that $f_i \in \mathcal{F}(U_i)$, i.e. that f_i has no poles on $U_i = \widehat{\mathbb{C}} - E_i$, i.e. that any poles of f_i lie in E_i. By the construction of the previous problem, $R_j(g_{ij})$ has no poles in E_j. But also $g_{ij} \in \mathcal{F}(U_{ij})$, meaning that all poles of g_{ij} lie in the union $E_i \cup E_j$. Thus the same must be true of g_{ij}, and since $R_j(g_{ij})$ moreover has no poles in E_j, we conclude that all poles of $R_j(g_{ij})$ lie on E_i. Passing to the sum, we conclude that all poles of f_i lie in E_i as desired.

We next show that $g_{ij} = f_i - f_j$. By our definition of f_i,

$$f_i - f_j = \sum_k R_k(g_{ik}) - \sum_\ell R_\ell(g_{j\ell}).$$

Since (g_{ij}) is a cocycle, we can replace g_{ik} in the first sum with $g_{ip} + g_{pk}$ for any fixed value of p. Doing so, we find

$$\sum_k R_k(g_{ik}) = \sum_k R_k(g_{ip}) + R_k(g_{pk}) = g_{ip} + \sum_k R_k(g_{pk});$$

the latter equality holding by problem (3). We can similarly replace the second sum, replacing the dummy index ℓ with k:

$$\sum_{\ell} R_\ell(g_{j\ell}) = g_{jp} + \sum_{k} R_k(g_{pk}),$$

Subtracting, we find $f_i - f_j = g_{ip} - g_{jp} = g_{ij}$, by one last application of the cocycle condition.