As usual, X denotes a compact Riemann surface of genus g.

(1) (a) By considering the short exact sequence
$$0 \to \mathcal{O} \to \mathcal{E}^0 \xrightarrow{\bar\partial} \mathcal{E}^{0,1} \to 0,$$
prove that $H^2(X, \mathcal{O}) = 0$.

(b) By considering the short exact sequence
$$0 \to \mathcal{O}_D \to \mathcal{O}_{D+P} \to \mathbb{C}_P \to 0,$$
prove that $H^2(X, \mathcal{O}_D) = 0$ for any divisor D.

(2) Let $\mathcal{M}$ denote the sheaf of meromorphic functions, and $\mathcal{Q}$ denote the sheaf of meromorphic 1-forms on X with zero residue at each pole.

(a) Let $f \in \mathcal{M}$ be given. Explain why $df \in \mathcal{Q}$ (i.e., why does the meromorphic 1-form df have zero residue at each pole?)

(b) Prove that
$$0 \to \mathbb{C} \to \mathcal{M} \xrightarrow{d} \mathcal{Q} \to 0$$
is exact.

(c) Convince yourself that the connecting map in sheaf cohomology $\delta : H^0(X; \mathcal{Q}) \to H^1(X; \mathbb{C})$ can be described concretely as the period map $\int : \mathcal{Q}(X) \to H^1(X; \mathbb{C})$. (Don't write anything down - once you understand the formalisms it really is just tautological. But do think about this!)

(d) Deduce that the period map $\mathcal{Q}(X) \to H^1(X; \mathbb{C})$ is surjective. You may use the fact that $H^1(X; \mathcal{M}) = 0$.

(3) Let X be the hyperelliptic curve defined by the equation $y^2 = x^5 - x$. Note that x and y are meromorphic functions on X. Compute the principal divisors $\operatorname{div}(x)$ and $\operatorname{div}(y)$. [Here, you should take X to be *compact*, by adding in point(s) at infinity. To check that your answer is correct, it might help to remember that every principal divisor has degree zero, and that the *topological* degree of a function is the degree of the positive part of the associated divisor.]

(4) Let X be the projective plane cubic defined by the equation $y^2 z = x^3 - xz^2$. Let $p_0 = [0 : 1 : 0]$, $p_1 = [0 : 0 : 1]$, $p_2 = [1 : 0 : 1]$, and $p_3 = [-1 : 0 : 1]$. Show that $p_1 + p_2 + p_3 \sim 3p_0$. [Hint: remember that a meromorphic function is the same thing as a holomorphic map to $\widehat{\mathbb{C}} = \mathbb{CP}^1$. Think about projection maps $\mathbb{CP}^2 \to \mathbb{CP}^1$ by forgetting a coordinate.]