As usual, X denotes a compact Riemann surface of genus g.

(1) (a) By considering the short exact sequence

$$0 \to \mathcal{O} \to \mathcal{E}^0 \xrightarrow{\overline{\partial}} \mathcal{E}^{0,1} \to 0,$$

prove that $H^2(X, \mathcal{O}) = 0$.

(b) By considering the short exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{O}_{D+P} \to \mathbb{C}_P \to 0,$$

prove that $H^2(X, \mathcal{O}_D) = 0$ for any divisor D.

Solution: For (a), we observe that the long exact sequence in sheaf cohomology includes the portion

$$H^1(X; \mathcal{E}^{0,1}) \to H^2(X; \mathcal{O}) \to H^2(X; \mathcal{E}^0).$$

The left and right terms vanish, since each of $\mathcal{E}^{0,1}$ and $\mathcal{E}^0$ are sheaves of $\mathcal{C}^\infty$ functions, which we showed to have vanishing higher cohomology in class.

For (b), we find the long exact sequence includes the portion

$$H^1(X; \mathbb{C}_P) \to H^2(X; \mathcal{O}_D) \to H^2(X; \mathcal{O}_{D+P}) \to H^2(X; \mathbb{C}_P).$$

We showed in class that $H^k(X; \mathbb{C}_P)$ for $k > 0$, so that the above establishes an isomorphism $H^2(X; \mathcal{O}_D) \cong H^2(X; \mathcal{O}_{D+P})$. As any divisor can be obtained from 0 by some finite sequence of addition and removal of points P, the result now follows from (a).

(2) Let $\mathcal{M}$ denote the sheaf of meromorphic functions, and $\mathcal{Q}$ denote the sheaf of meromorphic 1-forms on X with zero residue at each pole.

(a) Let $f \in \mathcal{M}$ be given. Explain why $df \in \mathcal{Q}$ (i.e., why does the meromorphic 1-form df have zero residue at each pole?)

(b) Prove that

$$0 \to \mathbb{C} \to \mathcal{M} \xrightarrow{d} \mathcal{Q} \to 0$$

is exact.

(c) Convince yourself that the connecting map in sheaf cohomology $\delta : H^0(X; \mathcal{Q}) \to H^1(X; \mathbb{C})$ can be described concretely as the period map $\int : \mathcal{Q}(X) \to H^1(X; \mathbb{C})$. (Don't write anything down - once you understand the formalisms it really is just tautological. But do think about this!)

(d) Deduce that the period map $\mathcal{Q}(X) \to H^1(X; \mathbb{C})$ is surjective. You may use the fact that $H^1(X; \mathcal{M}) = 0$.

Solution: For (a), let γ be a loop encircling a pole of f; we seek to show that $\int_\gamma df = 0$. By Stokes, $\int_\gamma df = \int_{\partial\gamma} f = 0$, since γ is a closed loop with empty boundary. For (b), exactness at $\mathbb{C}$ and $\mathcal{M}$ are obvious; we seek to show that locally at every point $p \in X$, the map of stalks $d : \mathcal{M}_p \to \mathcal{Q}_p$ is surjective. More concretely, choose a local coordinate centered at p, and suppose $g(z)dz \in \mathcal{Q}_p$ is given. If g is holomorphic at p then $f(z) = \int_p^z g(z)dz$ gives a local holomorphic antiderivative. If g has a pole at z, then choose some base point $q \neq p$ and define $f(z) = \int_q^z g(z)dz$ for z on some punctured neighborhood of p. *A priori* there is a concern that $f(z)$ could be ill-defined, but this is precluded by the hypothesis that the period of g at p is zero: a small punctured neighborhood of p has cyclic fundamental group, so every period is a multiple of the residue at p, zero by hypothesis.

Here are the gory details for (c). The connecting map δ can be described as follows. Let $\omega \in \mathcal{Q}$ be given, and let $\{U_i\}$ be an open covering of X that is sufficiently fine so that $\omega\,|_{U_i} = df_i$ has a local primitive f_i. Then $f_i - f_j \in \mathbb{C}$ determines a 1-cocycle for the sheaf $\mathbb{C}$. *A priori*, the target of the period map is a "different" $H^1(X;\mathbb{C})$: we view this here as the dual space to $H_1(X;\mathbb{C})$, and then the period map just integrates ω along the 1-cycle $\gamma \in H_1(X;\mathbb{C})$. We need to find a way to interpret this as giving a 1-cocycle in sheaf cohomology, i.e. a collection of $c_{ij} \in \mathbb{C}$ associated to intersections of sufficiently fine open coverings $\{U_i\}$. To do so, we suppose that the sets U_i and their intersections U_{ij} are simply-connected. Then any $\omega \in \mathcal{Q}(X)$ has a well-defined integral $f_i(z) = \int_{p_i}^z \omega$ defined on U_i, where $p_i \in U_i$ is an arbitrary base point. Note that different choices of p_i lead to primitives that differ by some constant. This leads us to define c_{ij} as the difference $c_{ij}(z) = f_i(z) - f_j(z) = \int_{p_i}^{p_j} \omega$ (yes, we may cross from one chart to the other in this definition, but the resulting integral is uniquely specified by analytic continuation).

We establish (d) by looking at the long exact sequence. By (c), the period map $\mathcal{Q}(X) \to H^1(X;\mathbb{C})$ can be interpreted as the connecting map $\delta : H^0(X;\mathcal{Q}) \to H^1(X;\mathcal{Q})$, and this is surjective because the next term in the sequence is $H^1(X;\mathcal{M})$ which we are told is zero.

(3) Let X be the hyperelliptic curve defined by the equation $y^2 = x^5 - x$. Note that x and y are meromorphic functions on X. Compute the principal divisors $\mathrm{div}(x)$ and $\mathrm{div}(y)$. [Here, you should take X to be *compact*, by adding in point(s) at infinity. To check that your answer is correct, it might help to remember that every principal divisor has degree zero, and that the *topological* degree of a function is the degree of the positive part of the associated divisor.]

Solution: By definition, the divisor of a meromorphic function is the formal sum of zeroes minus the formal sum of poles, each weighted by order. For (x), we see that the projection onto the x-coordinate has topological degree 2, so we are expecting two zeroes and two poles. The zeroes occur where $x = 0$; looking at the equation, there is a single solution $p_0 = (0,0)$ of multiplicity two. Since $x^5 - x$ has degree five, which is odd, we compactify X by adding a single point ∞, which then necessarily maps to $\infty \in \widehat{\mathbb{C}}$ with degree 2. We conclude

$$(x) = 2(0,0) - 2\infty.$$

The map y has degree five. We can see that $y = 0$ has the five solutions $x = 0$ and $x = e^{2\pi ik/4}$, each then necessarily of multiplicity one. There are no poles visible, meaning that y has a pole of order five at ∞:

$$(y) = (0,0) + (1,0) + (i,0) + (-1,0) + (-i,0) - 5\infty.$$

(4) Let X be the projective plane cubic defined by the equation $y^2z = x^3 - xz^2$. Let $p_0 = [0:1:0]$, $p_1 = [0:0:1]$, $p_2 = [1:0:1]$, and $p_3 = [-1:0:1]$. Show that $p_1 + p_2 + p_3 \sim 3p_0$. [Hint: remember that a meromorphic function is the same thing as a holomorphic map to $\widehat{\mathbb{C}} = \mathbb{CP}^1$. Think about projection maps $\mathbb{CP}^2 \to \mathbb{CP}^1$ by forgetting a coordinate.]

Solution: To show $p_1 + p_2 + p_3 \sim 3p_0$, we analyze the projection $\pi_x : [x:y:z] \mapsto [y:z]$. This is well-defined on X since the base point of the projection $[1:0:0]$ does not lie on X. In the affine coordinate patch given by $z = 1$, this projection map is just given by projecting $y^2 = x^3 - x$ onto the y coordinate; arguing as in the previous problem, we see this has divisor given as the sum of the roots with a triple pole at infinity. The roots are at p_1, p_2, p_3, while $\infty = [1:0]$ is mapped to only by $p_0 = [0:1:0]$ as desired. This constructs a meromorphic function f on X with divisor $(f) = p_1 + p_2 + p_3 - 3p_0$, so that $p_1 + p_2 + p_3 = 3p_0 + (f)$, showing the desired linear equivalence $p_1 + p_2 + p_3 \sim 3p_0$.