In problems 1-3, X is the hyperelliptic curve defined by $y^2 = p(x)$ for $p(x)$ a polynomial of degree $2g+2$ with distinct roots.

(1) Exhibit a basis for $\Omega(X)$ (global holomorphic differentials). As usual, each differential should be of the form $\omega = q(x,y)dx$ and/or $\omega = r(x,y)dy$, with $q(x,y), r(x,y)$ rational functions of x, y.

Solution: This is a variant of a problem we've done a lot of. As usual, the method is to identify a differential with no zeroes on the finite part of X; such a differential must have its zeroes hiding at the points at infinity. Since $\deg(p) = 2g+2$, X has genus g, and moreover has two points at infinity, say ∞_1 and ∞_2.

We start with the differential dx on $\mathbb{C}$ and pull back to X via the projection $\pi_x : \mathbb{C}^2 \to \mathbb{C}$. Since π_x locally has the expression $x \mapsto x^2$ at the $2g+2$ roots of p, we see that $\pi_x^*(dx)$ has simple zeroes at the $2g+2$ roots of p. These are cancelled by taking dx/y (suppressing the notation π_x^*), and we conclude that dx/y has no zeroes on the finite part of X.

We claim that $\omega_0 := dx/y$ is in fact holomorphic. As usual, we argue this by showing that ω_0 is an eigenvector for a symmetry that acts transitively on the points at infinity. Under the hyperelliptic involution $\iota : X \to X$ given by $(x,y) \mapsto (x,-y)$, we see that the two points at infinity are exchanged, while $\iota^*(\omega_0) = -dx/y = -\omega_0$. We conclude that $\mathrm{ord}_{\infty_1}(\omega_0) = \mathrm{ord}_{\infty_2}(\omega_0)$, and since ω_0 has no zeroes or poles outside of these two points, it follows that $\mathrm{ord}_{\infty_i}(\omega_0) = g-1$. In particular, $\omega_0 \in \Omega(X)$ is holomorphic.

To expand ω_0 into a basis for $\Omega(X)$, we consider expressions of the form $f(x)\omega(X)$ for $f(x)$ a polynomial. If $\deg(f) = d$, then f has d zeroes on $\mathbb{C}$ and hence has $2d$ zeroes on X. Thus $f(x)\omega_0$ must have a total of $2d$ poles at ∞_1 and ∞_2; arguing again by symmetry, we conclude that

$$\mathrm{ord}_{\infty_i}(f(x)\omega_0) = g-1-d.$$

Thus, $f(x)\omega_0$ is holomorphic so long as $\deg(f) \leqslant g-1$. We conclude that

$$\Omega(X) = \{f(x)dx/y \mid \deg(f) \leqslant g-1\},$$

which is a vector space of dimension g as required.

(2) Let $\iota : X \to X$ be the hyperelliptic involution $\iota(x,y) = (x,-y)$. Determine the eigenspaces of ι^* acting on $\Omega(X)$.

Solution: By the previous problem, any holomorphic differential on X has the form $\omega = f(x)dx/y$. Under the map $\iota(x,y) = (x,-y)$, this is clearly sent to $-\omega$, so every element of $\Omega(X)$ is a -1-eigenvector for ι.

(3) Let $P_0 \in X$ be a Weierstrass point, so that $P_0 = (x_0, 0)$ for some root x_0 of p, let $P' = (x,y)$ be any ordinary point (so that $y \neq 0$), and let P_∞ be one of the two points over ∞.

(a) Compute $h^0(nP_0)$ for all $n \geq 0$.

(b) Compute $h^0(nP')$ for all $n \geq 0$.

(c) Compute $h^0(nP_\infty)$ for all $n \geq 0$.

Solution: Since $p(x)$ has degree $2g+2$, X has genus g. By our new version of Riemann-Roch, for any $Q \in X$,

$$h^0(nQ) = 1 - g + n + h^0(K - nQ).$$

Thus to answer this question, we need to understand $h^0(K - nQ)$. This can be identified with the dimension of the space $H^0(\Omega_{-nQ})$ of holomorphic forms vanishing to order $\geq n$ at the point Q. In Problem 1, we saw that any holomorphic differential is of the form $f(x)dx/y$ for f a polynomial of degree at most $g-1$. If $Q = P_0$ is a Weierstrass point, then

$$\operatorname{ord}_{P_0}(f(x)dx/y) = 2\operatorname{ord}_{x(P_0)}(f),$$

since in local coordinates z near P_0, the projection π_x has the expression $x = z^2$.

We conclude that $H^0(\Omega_{-nP_0})$ is the space of differentials $f(x)dx/y$ where $\operatorname{ord}_{x(P_0)}(f) \geq n/2$, a space of dimension $g - \lceil n/2 \rceil$ (for $n \leqslant 2g-2$; it is empty for $n > 2g-2$). Plugging this in to Riemann-Roch, we find

$$h^0(nP_0) = 1 - g + n + (g - \lceil n/2 \rceil) = 1 + \lfloor n/2 \rfloor$$

for $n \leqslant 2g-2$, and

$$h^0(nP_0) = \begin{cases} 1 + \lfloor n/2 \rfloor & 0 \leqslant n \leqslant 2g-2 \\ 1 - g + n & n > 2g-2 \end{cases}$$

in general.

By contrast, let $Q = P'$ be an ordinary point. The difference in this case is that π_x is unbranched at P', so that

$$\operatorname{ord}_{P'}(f(x)dx/y) = \operatorname{ord}_{x(P')}(f).$$

Thus now $H^0(\Omega_{-nP'})$ is the space of differentials $f(x)dx/y$ with f vanishing to order $\geq n$ at $x(P')$, which has dimension $g - n$ for $n \leqslant g$ and 0 for $n > g$. In the first case, inserting into Riemann-Roch gives

$$h^0(nP') = 1 - g + n + (g - n) = 1,$$

while for $n > g$,

$$h^0(nP') = 1 - g + n,$$

so in total,

$$h^0(nP') = \begin{cases} 1 & n \leqslant g \\ 1 - g + n & n > g. \end{cases}$$

Finally, consider now $Q = P_\infty$, or in the notation previously established, (WLOG) $P_\infty = \infty_1$. By our previous analysis, we found that

$$\operatorname{ord}_{\infty_1}(f(x)dx/y) = g - 1 - \deg(f).$$

Thus $H^0(\Omega_{-n\infty_1})$ consists of such $f(x)dx/y$ for which $g-1-\deg(f) \geq n$, i.e. $\deg(f) \leqslant g-1-n$, a space of dimension $g-n$ for $n \leqslant g$ and trivial otherwise. Inserting this into Riemann-Roch, in the range $n \leqslant g$,

$$h^0(nP_\infty) = 1 - g + n + g - n = 1,$$

while for $n > g$,

$$h^0(nP_\infty) = 1 - g + n.$$

In total, then,

$$h^0(nP_\infty) = \begin{cases} 1 & n \leqslant g \\ 1 - g + n & n \geq g, \end{cases}$$

the same as for an ordinary point P'.

Comment. The message here is that Weierstrass points are exceptional: they have meromorphic functions of unusually low degree whose only pole is at P_0. One can define a Weierstrass point on an arbitrary Riemann surface as any point where the sequence $h^0(nP)$ is anything other than the generic behavior $h^0(nP) = 1$ for $n \leqslant g$. We will shortly see how to count the total number of Weierstrass points on any Riemann surface.

(4) Now let X be an arbitrary compact Riemann surface, and let $D = \sum_{i=1}^{d} P_i$ be a fiber of a nonconstant map $f : X \to \mathbb{CP}^1$ of the least possible degree $d > 0$.
   (a) Show that $\dim |D| = 1$. [Hint: If the dimension were any larger, construct a map $g : X \to \mathbb{CP}^1$ of smaller degree by a well-chosen projection.]
   (b) Let $E = K - D + P_1$. Show that $\dim |E| = g - d$, and that the base locus of E contains P_1.

Solution: Certainly $\dim |D| \geq 1$, since other fibers of f are in the linear system $|D|$. If $\dim |D|$ were at least 2, then one could obtain a map $g : X \to \mathbb{CP}^1$ of degree $d' < d$ as follows: take some linear system $L \subset |D|$ of dimension 2, and consider the associated map $\phi : X \to L^*$. This still realizes X as a subvariety of degree d. Now project from L^* onto $\mathbb{CP}^1$ from some point $P \in \phi(X)$. Since P is in the base locus of the pencil, it is not included in the divisor of the associated map $g : X \to \mathbb{CP}^1$. We conclude that g has degree $d' < d$, a contradiction.

For (b), we apply Riemann-Roch to the divisor $D - P_1$. Since $\deg(D - P_1) = d - 1$, we conclude that $h^0(D - P_1) = 1$, since otherwise this space would contain some nonconstant holomorphic map of degree strictly less than d. Riemann-Roch then says

$$h^0(D - P_1) = 1 - g + (d - 1) + h^0(K - D + P_1),$$

so that $h^0(K - D + P_1) = g - d + 1$. Since $\dim |E| = h^0(E) - 1$ for any divisor E, the claim follows. To see that moreover P_1 lies in the base locus of E, we recall that a point P lies in the base locus of $|E|$ if and only if $h^0(E) = h^0(E - P)$. We thus seek to show that $h^0(K - D + P_1) = h^0(K - D)$. Appealing once more to Riemann-Roch, we see that $h^0(D) = 2$ by part (a); this increase is offset by the increase of $\deg$ on the other side of the equation, showing that $h^0(K - D) = g - d + 1$ as well.

(5) Applying the preceding to a suitable hyperelliptic curve, give an explicit example of a complete linear system with nontrivial base locus and $\dim |E| > 0$.

Solution: We continue with the notation of the previous problem. For any hyperelliptic curve X, there is a nonconstant map $f : X \to \mathbb{CP}^1$ of degree $d = 2$. This is the smallest possible, since a map of degree 1 is necessarily an isomorphism. If X has genus $g \geq 2$, then by the previous problem, $\dim |E| = g - d > 0$ and contains P_1 in the base locus. To be more explicit, recall that $D = P_1 + P_2$ is a fiber of the degree-2 map $f : X \to \mathbb{CP}^1$. The linear system $K - D + P_1 = K - P_2$ then consists of canonical divisors that vanish at P_2. This problem is telling us that any ω vanishing at P_2 necessarily also vanishes at P_1.

This can in fact be seen another way, as follows. Recall that we constructed an explicit model for $\Omega(X)$ consisting of differentials $p(x)dx/y$, where p is a polynomial of degree at most $g - 1$. The map $f : X \to \mathbb{CP}^1$ of degree 2 sends $(x, y) \in X$ to x, so that if $P_1 = (x_0, y_0)$, then $P_2 = (x_0, -y_0)$. Said another way, $P_2 = \iota(P_1)$, where $\iota : X \to X$ sends (x, y) to $(x, -y)$ is the *hyperelliptic involution*. As we observed at the time, ι acts by -1 *on the entire space* $\Omega(X)$. Thus, ω vanishes at P_1 if and only if it vanishes at $\iota(P_1) = P_2$, which is what we observed above.