In this set, you will work through the following problem:

Let X be a non-hyperelliptic Riemann surface of genus 3, and let Y be a Riemann surface of genus 2. Then every holomorphic map $f : X \to Y$ is constant.

(1) An *involution* of a complex manifold Z is an element $\alpha \in \mathrm{Aut}(Z)$ of order 2.

  (a) Prove that the fixed-point set of any involution of $\mathbb{CP}^2$ contains a line (i.e. the image in $\mathbb{CP}^2$ of a 2-plane in $\mathbb{C}^3$).
  [Hint: $\mathrm{Aut}(\mathbb{CP}^2) = \mathrm{PGL}_3(\mathbb{C})$. Now think about linear algebra.]

  (b) Let X be a compact Riemann surface of genus 3 that is not hyperelliptic. Prove that every involution of X has a fixed point. [Recall: if a curve of genus 3 is not hyperelliptic, then the canonical map embeds it into $\mathbb{CP}^{g-1} = \mathbb{CP}^2$. Why does an automorphism of X extend to an automorphism of $\mathbb{CP}^2$?]

(2) Use Riemann-Hurwitz to prove that if $f : X \to Y$ is a nonconstant map, where X has genus 3 and Y has genus 2, then f has degree 2 and is unramified (has no branch points).

(3) Prove the statement at the top of the page. [You may want to brush up on the basic principles of covering space theory.]