

In this set, you will work through the following problem:

Let X be a non-hyperelliptic Riemann surface of genus 3, and let Y be a Riemann surface of genus 2. Then every holomorphic map $f : X \rightarrow Y$ is constant.

- (1) An *involution* of a complex manifold Z is an element $\alpha \in \text{Aut}(Z)$ of order 2.
- (a) Prove that the fixed-point set of any involution of \mathbb{CP}^2 contains a line (i.e. the image in \mathbb{CP}^2 of a 2-plane in \mathbb{C}^3).
[Hint: $\text{Aut}(\mathbb{CP}^2) = \text{PGL}_3(\mathbb{C})$. Now think about linear algebra.]
- (b) Let X be a compact Riemann surface of genus 3 that is not hyperelliptic. Prove that every involution of X has a fixed point. [Recall: if a curve of genus 3 is not hyperelliptic, then the canonical map embeds it into $\mathbb{CP}^{g-1} = \mathbb{CP}^2$. Why does an automorphism of X extend to an automorphism of \mathbb{CP}^2 ?]

Solution: Let $\alpha \in \text{PGL}_3(\mathbb{C})$ be an involution. Represent α by a matrix $A \in \text{GL}_3(\mathbb{C})$. Then $A^2 = \lambda I$ for some $\lambda \in \mathbb{C}^*$, but A itself is not of the form λI . In particular, A has at least two distinct eigenvalues, and if $A^2 = \lambda I$, then A must necessarily have exactly two eigenvalues $\pm\sqrt{\lambda}$. Moreover A must be diagonalizable, as any non-diagonalizable matrix (over \mathbb{C}) has infinite order. Since A has three eigenvalues in total, one of $\pm\sqrt{\lambda}$ must appear with multiplicity two. This eigenspace forms an A -invariant plane in \mathbb{C}^3 , which descends to the fixed line in \mathbb{CP}^2 .

Now let X be a non-hyperelliptic curve of genus 3. An involution α of X acts linearly on $\Omega(X)$ by the pullback α^* , inducing an involution of $H^0(X; \Omega)$ and hence an involution of $\mathbb{CP}^2 = \mathbb{P}H^0(X; \Omega)^*$. Thus α extends to an involution of the \mathbb{CP}^2 containing X in its canonical embedding. By the previous part of the problem, α fixes a line in \mathbb{CP}^2 . The points of intersection of X with this line are therefore fixed points of the original $\alpha : X \rightarrow X$.

- (2) Use Riemann-Hurwitz to prove that if $f : X \rightarrow Y$ is a nonconstant map, where X has genus 3 and Y has genus 2, then f has degree 2 and is unramified (has no branch points).

Solution: The Riemann-Hurwitz formula applied to a map $f : X \rightarrow Y$ yields

$$\begin{aligned}\chi(X) &= d\chi(Y) - \sum (d_y - 1) \\ -4 &= d(-2) - \sum (d_y - 1)\end{aligned}$$

Each ramification term d_y is an integer between 1 and d . In particular, $\sum (d_y - 1) > 0$, which forces $d \leq 2$, i.e. $d = 1$ or $d = 2$. Any holomorphic map of degree 1 is an isomorphism (since then every point would have ramification $d_y = 1$, hence there would be no critical points), so we conclude $d = 2$ as desired. In this case, the equality $-4 = d(-2)$ already holds, which forces $\sum (d_y - 1) = 0$. We conclude the

map is unramified as desired.

- (3) Prove the statement at the top of the page. [You may want to brush up on the basic principles of covering space theory.]

Solution: Suppose for contradiction that there is some nonconstant holomorphic map $f : X \rightarrow Y$. By the previous problem, f would be an unramified covering of degree 2. By covering space theory, such a cover is necessarily regular, so that f is induced by some deck transformation $\alpha : X \rightarrow X$ of order 2. By definition, deck transformations are fixed-point free. Thus, we have constructed a fixed-point-free involution of X , in contradiction with problem (1).