(1) Let

$$x^3y + y^3z + z^3x = 0$$

define the *Klein quartic* X, a smooth Riemann surface of genus 3 in $\mathbb{CP}^2$.

(a) Find nonzero complex numbers α, β, γ so that X admits an automorphism of the form $f([x : y : z]) = [\alpha x : \beta y : \gamma z]$ of order 7.
(b) Show that each fixed point of f is a Weierstrass point.
(c) Choose one such point P, and find an explicit meromorphic function on X with a pole just at P.

Solution: For (a), we observe that in order for the tuple α, β, γ to induce an automorphism of X, we must have that

$$\alpha^3\beta = \beta^3\gamma = \gamma^3\alpha.$$

We also observe that since we are working projectively, without loss of generality, we can set one of these (say, γ) to 1:

$$\alpha^3\beta = \beta^3 = \alpha.$$

In particular, these equations imply $\beta^{10} = \beta^3$, from which we conclude β (and hence α) are seventh roots of unity. Let $\zeta = e^{2\pi i/7}$. Then taking $\alpha = \zeta^3$ and $\beta = \zeta$ gives a solution to these equations.

For (b), we first note that the automorphism $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ given by $[x : y : z] \mapsto [\zeta^3 x : \zeta y : z]$ has exactly three fixed points, namely the points where two of the three coordinates vanish. (To see this, think back to last week's set: a fixed *point* of a projective automorphism is the same thing as a fixed *line* before projectivizing, i.e. an eigenvector, and the linear automorphism associated to f has three distinct eigenvalues by construction). By inspection, each of these three points lies on X. We claim that any one of these points is a Weierstrass point. Note that X is symmetric under permutation of the coordinates, so that it suffices to consider any one particular point. We will investigate the point $[0 : 0 : 1]$. Working in affine coordinates $z = 1$, X is given by

$$x^3y + y^3 + x = 0.$$

We compute the tangent line through $(0, 0)$ by computing partial derivatives:

$$f_x = 3x^2y + 1 \quad \text{and} \quad f_y = x^3 + 3y^2,$$

so that the normal line at $(0, 0)$ is given by $\nabla f = (1, 0)$ and hence the tangent line is given by $x = 0$. We see that this intersects X with multiplicity 3 at $(0, 0)$ (the system of equations reduces to $y^3 = 0$), so that $(0, 0)$ is a flex and hence a Weierstrass point.

To produce an explicit meromorphic function, the idea is to take the flex line L meeting X at $3P + Q$, and project from Q. Sending L to ∞ via such a pencil map will produce a meromorphic function with a triple pole at P and nowhere else. Back in $\mathbb{CP}^2$, the tangent line L is still given by $x = 0$, which meets the projective curve X at $Q = [0 : 1 : 0]$. A second line through this point is given by $z = 0$. Thus the meromorphic function $\phi([x : y : z]) = z/x$ will be of the desired form.

(2) Let D be an effective divisor on a compact Riemann surface X. Prove that $\dim |D| \geq t$ if and only if for every t points $P_1, \ldots, P_t$ on X, there is $D' \in |D|$ passing through $P_1, \ldots, P_t$.

Solution: Recall that $|D|$ is a projective space of dimension $h^0(D) - 1$, equipped with a map $\phi_D : X \to |D|$, and the effective divisors $D' \in |D|$ arise as hyperplane sections $\phi_D(X) \cap H$. This map is *nondegenerate* in the sense that $\phi_D(X)$ does not lie in any hyperplane. In $\mathbb{CP}^n$, any t points lie in some hyperplane so long as $t \leqslant n$, and conversely, if $t > n$, then, since the image is nondegenerate, there is some collection of t points in $\phi_D(X)$ not lying on any hyperplane. The claim follows.

(3) Let D, E be effective divisors. Prove that

$$\dim |D| + \dim |E| \leqslant \dim |D + E| .$$

[Hint: Use the previous problem. Riemann-Roch will get you nowhere.]

Solution: We use the previous problem. We seek to show that given any $t = \dim |D| + \dim |E|$ points $\{P_i\}$ on X, there is a divisor in $|D + E|$ passing through $\{P_i\}$. Divide these up, finding a divisor $D' \in |D|$ passing through the first $\dim |D|$ and a divisor $E' \in |E|$ passing through the remaining points; the divisor $D' + E'$ is effective and lies in the linear system $|D + E|$, as required.

(4) (Clifford's theorem) Let D be a special divisor of degree d. Prove that $\dim |D| \leqslant \frac{d}{2}$. [Hint: apply the preceding to the pair of divisors $D, K - D$ and then invoke Riemann-Roch. Be sure to explain where the hypothesis that D is special is being used.]

Solution: Since D is special, there is some effective canonical divisor K containing D, so that $K - D$ is effective. Then by the previous problem, taking $E = K - D$, we find that

$$\dim |D| + \dim |K - D| \leqslant \dim |K| = g - 1.$$

By Riemann-Roch,

$$\dim |D| = d - g + 1 + \dim |K - D| ,$$

so that

$$\dim |D| - \dim |K - D| = d - g + 1.$$

Adding these together,

$$2 \dim |D| \leqslant d,$$

as required.