Instructions:. This is a take-home exam due by 11:59 pm on Wednesday May 7 to my email (nsalter@nd.edu). You may consult outside resources, but document *everything* that you use (apart from your own notes from class). You may not work with or discuss this with anyone else, and all work that you submit must reflect only your own understanding.

I *strongly encourage you* to type your solutions up. The exam will be graded both for correctness and for clarity of your expositions.

(1) Given any two meromorphic functions f, g, show that there is a divisor D such that both f and g are in $\mathcal{O}_D$.

(2) Do not use Riemann-Roch for this problem. Suppose that X is a compact Riemann surface and $D > 0$ is a strictly positive divisor such that $h^0(D) = 1 + \deg(D)$. Show that there is a point $p \in X$ such that $h^0(p) = 2$. Conclude that $X \cong \mathbb{CP}^1$.

(3) Show that given any point p on X, there is a global meromorphic 1-form ω on X with a double pole at p and no other poles.

(4) Consider the triple branched covering X of $\mathbb{CP}^1$ defined by $y^3 = x^6 - 1$.

(a) Show that X has genus 4.

(b) Find a basis for $\Omega(X)$.

(c) Show that the canonical embedding $X \to \mathbb{CP}^3$ can be given in affine coordinates as $\phi(x, y) = (x, x^2, y)$.

(d) Find (affine) quadric and cubic hypersurfaces that contain (the affine part of) $\phi(X)$. (Explicitly, just find equations in x, y of degree 2 or 3 containing the points of the form (x, x^2, y). Since you are working affinely, the equations need not be homogeneous).