Instructions:. This is a take-home exam due by 11:59 pm on Thursday May 7 to my email (nsalter@nd.edu). You may consult outside resources, but document *everything* that you use (apart from your own notes from class). You may not work with or discuss this with anyone else (including AI), and all work that you submit must reflect only your own understanding. The exam will be graded both for correctness and for clarity of your expositions.

Part I: independent reading. For this first part, you will read about a topic on your own and prepare a short digest of what you learned.

Topic options:

(1) **The Hodge theorem.** For the analytically inclined. Read Chapter 0.6 of Griffiths-Harris (pages 80-106), or another source of your choice.

(2) **The Torelli theorem.** It turns out that a Riemann surface can be *recovered* from its Jacobian, once it has been endowed with some additional data called a "principal polarization". Read an outline in McMullen (Ch. 14, pp. 130-131, 133-136), then read the argument in GH. The main argument is given on p. 359-362; there is a long discussion of line bundles on Abelian varieties, polarizations (their Chern classes), and theta functions (holomorphic sections of these line bundles) that occurs prior to this in sections 2.6 and 2.7. This should be skimmed (especially the sections on line bundles on p. 307-317 and theta functions on p. 317-321, as well as Riemann's theorem (p. 333-340), but the proof of Torelli itself is relatively short if you black-box Riemann's theorem.

(3) **Resolving plane curve singularities.** Riemann surfaces arise from solutions to polynomials that are everywhere *smooth*. What can be done when this fails? This is the subject of *singularity theory*. A basic tool is to *resolve the singularity*, by finding a Riemann surface together with an algebraic map from the Riemann surface to the singular curve. There are various approaches to this, which are presented in Chapter 8 of *Plane algebraic curves* by Brieskorn-Knorrer. Chapter 8 spans 250 pages (!!!) of the book, but this is because there are many many explicit examples, chatty discussions, and historical interludes. There are two key methods - Puiseux series (p. 370-455) and blowing up, aka "quadratic transformations" (p. 455-535), but you'll be able to get the gist without reading every single word. Read about one, the other, or both.

(4) **Dealer's choice.** If there's a topic connected to Riemann surfaces you've been interested in, let me know and we can discuss some sources. **If you want to go this route, let me know by email no later than Wednesday, April 29. I won't accept this option if you don't clear it with me by this deadline.**

If you need help accessing any of the suggested books, or navigating the reading outlines I suggested, contact me by email.

What do I need to hand in for this part? Hand in a summary of what you have read *of no more than one page*. The goal here is to get some practice summarizing and presenting longer, more complicated arguments. As such, I'd recommend making an explicit outline (with steps, substeps, etc.) of the theorems you studied, as if you were going to be presenting this to a friend, your advisor, or your students. Try to break what you learned into two to four (give or take) major steps, with a few substeps each, and give one to two sentences of explanation of the key ideas for each of the substeps.

Part II: problems. As usual, X denotes a compact Riemann surface of genus g.

(1) Let $d \geq 1$ be fixed. Prove that there is an embedding $\phi : X \to \mathbb{CP}^N$ such that if D is *any* effective divisor of degree d, then there is a projection $\pi_D : \mathbb{CP}^N \dashrightarrow |D|^*$ such that the composition $\pi_D \circ \phi : X \to |D|^*$ is the map associated to D.

(2) Show that given any point p on X, there is a global meromorphic 1-form ω on X with a double pole at p and no other poles. (Yes, this follows from Mittag-Leffler, but don't just invoke this - explain how this follows from Riemann-Roch).

(3) Let $X \subset \mathbb{CP}^2$ be a smooth cubic plane curve. Let $\infty \in X$ be a flex point (a point where the tangent line makes contact of order 3), and endow X with the usual group structure ("collinear triples sum to zero"), taking ∞ as the identity element.
   (a) Prove that the Abel-Jacobi map $\lambda : (X, \infty) \to (\mathbb{C}/\Lambda, 0)$ is compatible with the Abelian group structures on the domain and codomain.
   (b) Let $P \in X$ be a nontrivial 2-torsion point. Among all the tangent lines L_Q to points $Q \in X$, how many pass through P?
   (c) How many distinct flexes are there on X?
   (d) Suppose $L \in \mathbb{CP}^2$ passes through two flex points of X. Prove that L then passes through a third flex.

(4) Consider the triple branched covering X of $\mathbb{CP}^1$ defined by $y^3 = x^6 - 1$.
   (a) Determine the genus of X.

   (b) Find a basis for $\Omega(X)$.

   (c) Show that the canonical embedding $X \to \mathbb{CP}^3$ can be given in affine coordinates as $\phi(x, y) = (x, x^2, y)$.

   (d) Find (affine) quadric and cubic hypersurfaces that contain (the affine part of) $\phi(X)$. (Explicitly, just find equations in x, y of degree 2 or 3 containing the points of the form (x, x^2, y). Since you are working affinely, the equations need not be homogeneous).