Lemma (Uniqueness of branched covers)

X, Y be cpt RS., $f : X \to \hat{\mathbb{C}}$, $g : Y \to \hat{\mathbb{C}}$ hol.

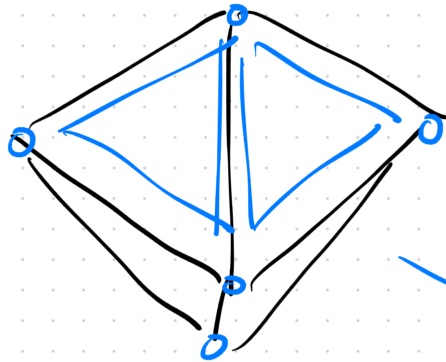
Suppose ① branch sets $B(f) = B(g) = \underline{B} \subseteq \hat{\mathbb{C}}$ (crit. vals) are equal

② covering spaces $X^\circ \to \hat{\mathbb{C}} - B$, $Y^\circ \to \hat{\mathbb{C}} - B$ are iso. as covering spaces

$$\left(\begin{array}{c} X^\circ \longrightarrow Y^\circ \\ f \searrow \ \hat{\mathbb{C}} \ \swarrow g \end{array} \quad \text{"covers the identity)} \right)$$

Then $X \cong Y$ are iso. as R.S.

Example



$\mathbb{O}$: octahedron.

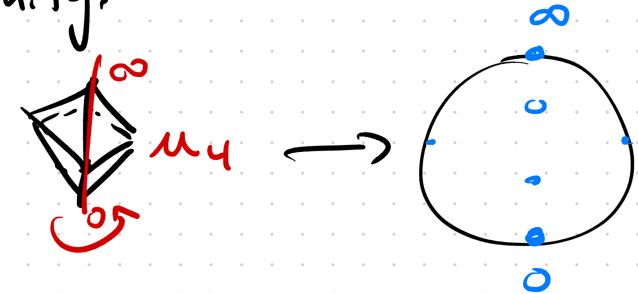
Carries RS structure.

Uniformization theorem: Let X be a cpt RS. of genus 0. Then $X \cong \hat{\mathbb{C}}$ as RS.

Claim: $\exists$ uniform. map $f: \mathbb{O} \to \hat{\mathbb{C}}$ s.t. vertices go to $0, \infty$, 4^{th} rts of unity.

Claim: $\exists$ uniform. map $f: \mathbb{D} \to \hat{\mathbb{C}}$ s.t. vertices go to $0, \infty$, 4^{th} rts of unity.

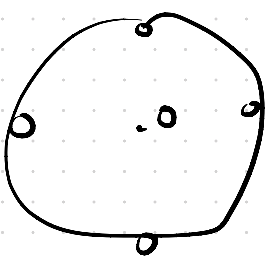
Pf| $\exists$ aut $\alpha: \mathbb{D} \to \mathbb{D}$:



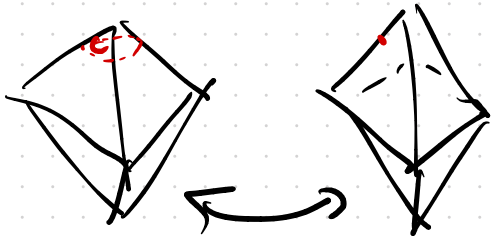
Fix ∞: $az + \cancel{b} = 0$

$\Rightarrow$ a must be 4^{th} rt. of 1.

Normalize to send remaining 4 to μ_4.



$X :=$ double branch cover of $\mathbb{O}$ at vertices.



$\pi_1(\mathbb{O} - \text{Verts}) \to \mathbb{Z}/2$

$Y := \; y^2 = x(x^4-1) \qquad (x,y) \longleftrightarrow (x,-y)$

R.H.: X: $\quad 2-2g_X = 2(2) - 6\cdot(2-1) = -2$

$g_X = 2.$

$\overline{Y} \to \hat{\mathbb{C}}$: branched at $0, \mu_4, \infty$

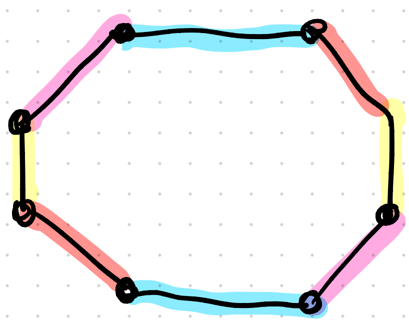
(since deg is 5)

Observe: under ident. $\mathbb{D} \to \hat{\mathbb{C}}$, double cover $X \to \mathbb{D}$ is branched at $0, \mu_4, \infty$. As is Y.

- $B(X) = B(Y) \subseteq \hat{\mathbb{C}}$.
- Also $X, Y \cong$ as covering spcs:

each is topol. classified by $\pi_1(S^2 - 6\,\text{pts}) \to \mathbb{Z}/2$

each gen $\longmapsto 1$



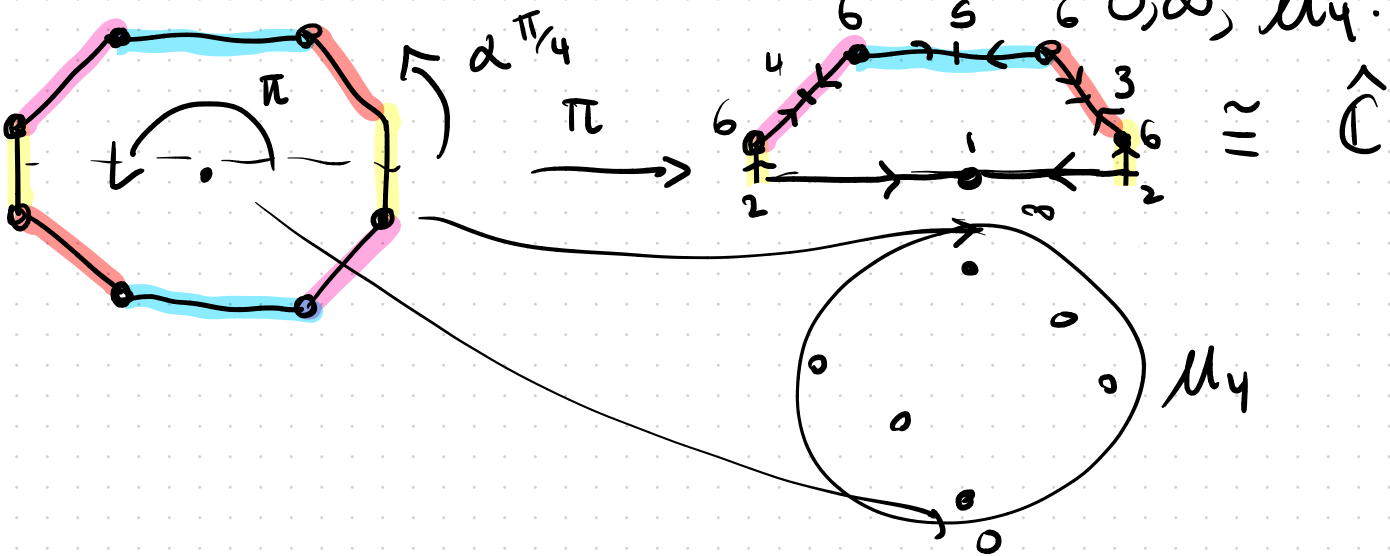
Z = octagon w. opp faces identified.

Claim: Z is a R.S.

$$\chi = V - E + F = -2$$
$$1 - 4 + 1$$

Claim: $Z \cong X \cong Y$.

Pf: Again realize Z as $Z \to \hat{\mathbb{C}}$ branched over $0, \infty, \mu_4$.



Belyi's Theorem

Number field: a finite extension of $\mathbb{Q}$: $K/\mathbb{Q}$

K is fin. dim'l as VS $/\mathbb{Q}$.

(Solutions to polys w. $\mathbb{Q}$-coeffs.)

eg $\mathbb{Q}(\sqrt{2})$

Defn X a cpt RS is **defined over K** (# field)

if X is obtained as the closure of solutions to polys w. coeffs in K.

Theorem (Belyi) X cpt RS is defined / # field if and only if

$\exists \; f: X \to \hat{\mathbb{C}}$ w. branch set $B(f) \subseteq \{0, 1, \infty\}$.

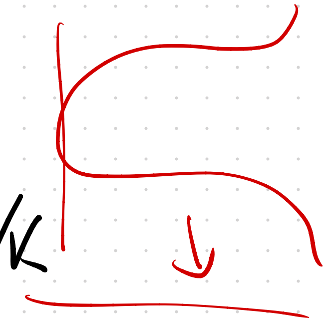
<u>Pf</u> Sps X defined / K.

X is given as closure of $V(F(x,y)) \subseteq \mathbb{C}^2$, with $F \in K[x,y]$.

Define $\pi: X \to \hat{\mathbb{C}}$ via $(x,y) \mapsto x$

Crit pts: where $F_y = 0$.

Solutions to $F = F_y = 0$ lie in some ext. L/K



Therefore $B(\pi) \subseteq L \subseteq \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$

Strategy: Construct $g: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ s.t.

$$X \xrightarrow{\pi} \hat{\mathbb{C}} \xrightarrow{g} \hat{\mathbb{C}} \text{ has } B(g \circ \pi) \subseteq \{0,1,\infty\}$$

Step 1: Reduce the degree of points in $B(\pi)$

$z \in K$: $\deg(z) = \min \{\deg(f(T)) : f \in \mathbb{Q}[T], f(z)\}$

$\deg(z) = 1 \iff z \in \mathbb{Q}$.

Choose $z \in B(\pi)$ of max'l deg:

z s.t. some poly. $P \in \mathbb{Q}[T]$ of deg d.

Q'n: what is $B(P)$? Crit pts lie at rts of P'.

P' has deg $d-1 <$ max deg. $\Rightarrow$ $B(P) \subseteq$ # field of deg $< d$.

$$B(P \circ \pi) = B(P) \cup P(B(\pi))$$

$P(z) = 0$

Step 1 ✓.

Step 2: $B(\pi) \subseteq \mathbb{Q}$.

Consider $P(z) = C \cdot z^a (z-1)^b$ $(a, b \in \mathbb{N})$

$C \in \mathbb{Q}$

crit pts: $\{0, 1, \omega = a/(a+b)\}$.

Choosing C carefully: $P(\omega) = 1$. $\Rightarrow B(P) \in \{0, 1, \infty\}$.

Take $x \in B(\pi) \subseteq \mathbb{Q}$. If nec., apply some Möbius transformation so that $x \in (0, 1)$. Then,

$$B(P \circ \pi) = \underbrace{B(P)}_{\{0,1,\infty\}} \cup \underbrace{P(B(\pi))}_{\text{has one fewer point outside } \{0,1,\infty\}}.$$

$\Rightarrow$ ✓

<u>Conversely</u>: Suppose $f: X \to \hat{\mathbb{C}}$ br. over $0, 1, \infty$.

<u>Nontrivial fact</u>: $\exists\ g: X \to \hat{\mathbb{C}}$ s.t.

$(f, g): X^\circ \to \mathbb{C}^2$ is an <u>immersion</u>, $X^\circ \subseteq \mathbb{C}^2$, deg d.

s.t. $X^\circ = V(P_0)$ for $P_0 \in \mathbb{C}[X, Y]_{(d)}$

Want to find a suitable $P \in K[X, Y]_{(d)}$,

s.t. $X^\circ \cong V(P)$.

<u>Key obs</u>: For any $P \in \mathbb{C}[X, Y]_{(d)}$, and any $z \in \mathbb{C}$,

the condition that $V(P)$ branches over z is

a polynomial on the vector spc $\mathbb{C}[X, Y]_{(d)}$

eg: $ax^2+bxy+cy^2$, $z \in \mathbb{C}$

$$\begin{array}{ccc} \downarrow & (x,y) & \\ & \rotatebox{90}{$\mapsto$} & \\ \mathbb{C}_x & x & x=z\ ? \end{array}$$

Solns to $ax^2+bxy+cy^2=P$, $P_y=0$

$P_y = bx+2cy \rightsquigarrow y=\ldots$ eqn of a,b,c,x

$z=x$: eqn only in a,b,c

Upshot: eqns on $\mathbb{C}[x,y](\omega)$ saying that X° branches over $0,1,(\infty)$ is a poly on with rational coeffs.

$\Longrightarrow\ \exists$ solutions over $\overline{\mathbb{Q}} \Longrightarrow$ i.e. K.

Claim any solution is $\cong$ to X° via uniqueness.