Instructions: Do not discuss these problems with anyone else, and do not use any generative AI tools. Otherwise, you may consult any references you like as long as you document what you use. Solutions will be graded both for correctness and for clarity of exposition.

Submission instructions: Email your solutions to me (*not* our grader!) at nsalter@nd.edu before midnight, March 6.

(1) Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disk. Let $X \cong S^2$ be the compact topological surface obtained from Δ by taking the one-point compactification. Prove that there is no Riemann surface structure on X that includes Δ as a chart (with coordinate function given by the inclusion $\Delta \subset \mathbb{C}$).

In problems 2-4, X denotes the solution set in $\mathbb{CP}^2$ to the equation

$$F(x, y, z) = x^4 + y^4 + z^4 + x^2 z^2.$$

(2) Verify that X is a Riemann surface.

(3) By considering a convenient projection $\pi : X \subset \mathbb{CP}^2 \to \mathbb{CP}^1$, use the Riemann-Hurwitz formula to show that X has genus 3.

(4) Construct a basis of explicit holomorphic differentials for $\Omega(X)$.

(5) (a) Prove that divisors D, E on $X = \mathbb{CP}^1$ are linearly equivalent if and only if $\deg(D) = \deg(E)$.
(b) Show that the above statement is false when X is replaced by any compact Riemann surface of positive genus.

(6) Let $\mathcal{O}^*$ denote the sheaf of *nonvanishing* holomorphic functions on a compact Riemann surface X. Prove that $H^2(X, \mathcal{O}^*) = 0$.