Instructions: Do not discuss these problems with anyone else, and do not use any generative AI tools. Otherwise, you may consult any references you like as long as you document what you use. Solutions will be graded both for correctness and for clarity of exposition.

(1) Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disk. Let $X \cong S^2$ be the compact topological surface obtained from Δ by taking the one-point compactification. Prove that there is no Riemann surface structure on X that includes Δ as a chart (with coordinate function given by the inclusion $\Delta \subset \mathbb{C}$).

Solution: Let $\infty \in X$ denote the point added to Δ in the one-point compactification. Suppose such a structure exists, and let $U \subset X$ be a coordinate neighborhood of ∞ with coordinate function $\phi : U \to \mathbb{C}$; assume $\phi(\infty) = 0$. Note that $U \cap \Delta = U \setminus \{\infty\}$; let $f : \phi(U) \setminus \{0\} \to \Delta$ be the ostensibly holomorphic transition function.

This has an isolated singularity at 0. Since $\mathrm{Im}(f) \subset \Delta$ is bounded, this must be a *removable* singularity, and so f extends to a holomorphic map $F : \phi(U) \to \overline{\Delta}$, where $\overline{\Delta} = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$. But this is nonsense: f realizes a homeomorphism between a punctured neighborhood of $0 \in U$ and a neighborhood of the boundary in Δ. Thus f does not even admit a *continuous* extension over 0 (since e.g. one can construct sequences $\{a_i\}, \{b_i\}$ limiting in to $0 \in U$ with different limits $\lim_{i\to\infty} f(a_i) \neq \lim_{i\to\infty} f(b_i)$ on the boundary of Δ).

In problems 2-4, X denotes the solution set in $\mathbb{CP}^2$ to the equation

$$F(x, y, z) = x^4 + y^4 + z^4 + x^2z^2.$$

(2) Verify that X is a Riemann surface.

Solution: We will examine the affine equations defining X and verify that there is no point where both partial derivatives vanish. In the chart $z = 1$, the equation is $f(x, y) = x^4 + x^2 + y^4 + 1 = 0$, with partial derivatives $f_x = 4x^3 + 2x$ and $f_y = 4y^3$. If $f_y = 0$ then $y = 0$ and so $x^4 + x^2 + 1 = 0$. If also $f_x = 0$, then $4x^3 + 2x = 0$, and it can be seen that these latter equations have no common solution. Looking the equation in the chart $x = 1$ gives $f = 1 + y^4 + z^4 + z^2$, with partials $f_y = 4y^3$ and $f_z = 4z^3 + 2z$. Similarly to before, the condition $f_y = 0$ leads to $z^4 + z^2 + 1 = 0$ and $4z^3 + 2z = 0$, which again has no common solution. This shows that X is smooth everywhere at least one of x, z is nonzero, but the point $[0 : 1 : 0]$ does not lie on X, so we are done.

(3) By considering a convenient projection $\pi : X \subset \mathbb{CP}^2 \to \mathbb{CP}^1$, use the Riemann-Hurwitz formula to show that X has genus 3.

Solution: We work with the projection p given in the x, y affine plane as $p(x, y) = x$. We first note that p has degree 4, since fixing a generic x_0, the polynomial $f(x_0, y)$ has degree 4 in y. This line of though also shows that f is branched wherever the polynomial $f(x_0, y) = y^4 + (x^4 + x^2 + 1)$ has fewer than four roots, i.e. at the roots of $p(x) = x^4 + x^2 + 1$. It's important to note that there are four *distinct* roots to $p(x)$, hence four branch points - to see this note that setting $t = x^2$ expresses p as a quadratic in t with distinct nonzero roots, so that p does indeed have four distinct roots. At one of these points, the number of solutions to $y^4 = p(x)$ drops from 4 to 1, showing that each point has ramification index $d_x = 4$. We must also consider the possibility of ramification over ∞, i.e. points with $z = 0$. There are 4 such points: the solutions to $x^4 + y^4 = 0$, and we conclude that p is unramified over ∞. Applying Riemann-Hurwitz then gives

$$2 - 2g = \chi(X) = 4 \cdot 2 - 4(4-1) = 8 - 12 = -4,$$

showing $g = 3$ as required.

(4) Construct a basis of explicit holomorphic differentials for $\Omega(X)$.

Solution: Continuing with the setup of the previous problem, we pull back the meromorphic differential dx from $\mathbb{CP}^1$ to X. The pullback $p^*(dx)$ acquires a zero of order $d_x - 1$ at a branch point of order d_x. We conclude that dx has zeroes of order 3 at each of the four branch points (equivalently, the points with $y = 0$). Since the total sum of zeroes and poles must be $2g - 2 = 4$, we conclude that dx has a total of 8 poles hiding at the four points at infinity. Since dx has a pole of order 2 at infinity and each of the four points over infinity is unramified, we conclude that $p^*(dx)$ acquires a double pole at each of these points.

Since the function y has a simple pole at infinity, it follows that $\omega_1 = p^*(dx)/y^2$ is holomorphic at the points at infinity. On the other hand, this introduces double poles at the points with $y = 0$. But since $p^*(dx)$ has triple zeroes at $y = 0$, the resulting ω_1 is a holomorphic form with simple zeroes at the four points with $y = 0$ and nowhere else, as to be expected.

Since $g(X) = 3 = \dim(\Omega(X))$, we need to find two more forms. One obvious choice is $\omega_2 = p^*(dx)/y^3$: this now is nonvanishing at the points $y = 0$, and introduces simple zeroes at the four points at infinity. Finally, we claim $\omega_3 = xp^*(dx)/y^3$ is holomorphic: x has a simple pole at ∞ which cancels the simple zeroes at infinity of ω_2, and introduces zeroes at the four points $x = 0$. Clearly $\omega_1, \omega_2, \omega_3$ are linearly-independent, since they were obtained from one another by multiplying by the non-constant functions x, y.

(5) (a) Prove that divisors D, E on $X = \mathbb{CP}^1$ are linearly equivalent if and only if $\deg(D) = \deg(E)$.
(b) Show that the above statement is false when X is replaced by any compact Riemann surface of positive genus.

Solution: (a) By definition, $D \sim E$ if there is a meromorphic function $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ with $(f) = D - E$. Recall that the degree of any principal divisor is zero: $\deg((f)) = 0$, showing that $\deg(D) = \deg(E)$ is a *necessary* condition. It remains to show that for $X = \mathbb{CP}^1$, this is sufficient.

Observe that linear equivalence is a condition only on the difference $D - E$, and so $D \sim E$ if and only if $D + F \sim E + F$ for any auxiliary divisor F. In particular, we may assume that each point $P \in \mathbb{CP}^1$ is contained in the support of at most one of D, E. Moreover, writing $D = D^+ - D^-$ with $D^\pm \geq 0$ of disjoint support, and likewise writing $E = E^+ - E^-$ decomposed similarly, we conclude that $D \sim E$ if and only if $D^+ + E^- \sim E^+ + D^-$. To summarize, we have reduced to the special case that D and E are effective and have disjoint support.

Write

$$D = \sum a_P P \quad \text{and} \quad E = \sum b_Q Q$$

with each of $a_P, b_Q \geq 0$. By hypothesis, $\sum a_P = \sum b_Q := d$. Without loss of generality, suppose that ∞ is *not* in the support of D (it may or may not be in the support of E).

Define the polynomial functions

$$p(z) = \prod (z - P)^{a_P} \quad \text{and} \quad q(z) = \prod_{Q \neq \infty} (z - Q)^{b_Q},$$

and compute that

$$(p) = D - d\infty \quad \text{and} \quad (q) = (E - b_\infty \infty) - (d - b_\infty)\infty = E - d\infty.$$

Define $f = p/q$, and then compute

$$(f) = (p) - (q) = D - E,$$

realizing $D - E$ as a principal divisor as desired.

For (b), let $g(X) > 0$, and let $P \neq Q$ be distinct points. We claim that $P \not\sim Q$. If this were the case, there would be a meromorphic function $f : X \to \mathbb{CP}^1$ with $(f) = P - Q$. In particular, $\deg(f) = 1$ and f would be an unbranched covering of $\mathbb{CP}^1$ of degree 1, i.e. an isomorphism.

(6) Let $\mathcal{O}^*$ denote the sheaf of *nonvanishing* holomorphic functions on a compact Riemann surface X. Prove that $H^2(X, \mathcal{O}^*) = 0$.

Solution: We consider the exponential short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1.$$

Taking the long exact sequence in cohomology yields the portion

$$\cdots \to H^2(X, \mathcal{O}) \to H^2(X, \mathcal{O}^*) \to H^3(X, \mathbb{Z}) \to \ldots$$

We proved on Homework 6 that $H^2(X, \mathcal{O}) = 0$. Since $H^3(X, \mathbb{Z})$ is just ordinary cohomology, and since X is a Riemann surface and *a fortiori* a manifold of dimension 2, also $H^3(X, \mathbb{Z}) = 0$. This traps $H^2(X, \mathcal{O}^*)$ in between two vanishing terms in an exact sequence, showing vanishing.